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THE FIVE-DIMENSIONAL GEOMETRY OF THE CURVATURE TENSOR IN A RIEMANNIAN V_4

By H. S. RUSE (*Southampton*)

[Received 27 February 1944]

1. Introduction

IF X^i is a contravariant vector at a given non-singular point P, (x^i), of a Riemannian V_n of fundamental tensor g_{ij}, then the X^i may be interpreted as the homogeneous coordinates of a point in the projective $(n-1)$-space S_{n-1} at infinity in the centred affine tangent-space T_n at P. If

$$p^{ij} = X^iY^j - X^jY^i$$

are Plücker coordinates of the line joining points X^i, Y^i of S_{n-1} (the p^{ij} therefore being the components of a simple bivector of V_n at P), then the lines for which

$$R_{ijkl}p^{ij}p^{kl} = 0, \tag{1.1}$$

where R_{ijkl} is the covariant Riemann tensor (skew in i, j and in k, l), form a quadratic complex in S_{n-1}, the *Riemann complex*.

In a previous paper (Ruse, **6**) it was shown that, for $n = 3$, the theory of the Riemann complex reduces to a simple one of conics in a projective S_2, and an introductory discussion was given of the case $n = 4$. It is the purpose of this paper to continue that discussion. Hereafter all Latin indices will run from 1 to 4.

For $n = 4$ the theory is one of a quadratic complex in a projective S_3, together with a non-degenerate quadric, the *fundamental quadric*, of point-equation $g_{ij}X^iX^j = 0$, of tangential equation $g^{ij}u_iu_j = 0$, and of line-equation

$$g_{ijkl}p^{ij}p^{kl} = 0, \quad \text{or} \quad g^{ijkl}\,{}^\circ p_{ij}{}^\circ p_{kl} = 0, \tag{1.2}$$

where

$$g_{ijkl} \equiv g_{ik}g_{jl} - g_{il}g_{jk}, \tag{1.3}$$

and where ${}^\circ p_{ij}$ are the dual coordinates of the line p^{ij}, defined by

$${}^\circ p_{ij} = \tfrac{1}{2}\epsilon_{ijkl}p^{kl}. \tag{1.4}$$

Here the $\frac{1}{2}$ is a factor of proportionality chosen for convenience, and ϵ_{ijkl} is the dualizing tensor, or *dualizor*, of components $\pm\sqrt{g}$, 0. The dualizors ϵ_{ijkl}, ϵ^{ijkl} $(= \pm 1/\sqrt{g}, 0)$ are imaginary if P is a point of V_4 at which $g < 0$. When the fundamental quadric is taken in either of the forms (1.2), it is to be regarded as defined by the quadratic

complex of lines touching it, (1.2) being dual equations of this special complex because

$$\begin{aligned} {}^\circ g^{ijkl} &\equiv \tfrac{1}{4}\epsilon^{ijmn}\epsilon^{klpq}g_{mnpq} \\ &= g^{ijkl}, \end{aligned} \tag{1.5}$$

by Jacobi's theorem applied to the second-order minors of the adjugate of the determinant $|g_{ij}|$.

The dual equation of the Riemann complex is

$$^\circ R^{ijkl}\,{}^\circ p_{ij}\,{}^\circ p_{kl} = 0, \tag{1.6}$$

where

$$^\circ R^{ijkl} \equiv \tfrac{1}{4}\epsilon^{ijmn}\epsilon^{klpq}R_{mnpq}. \tag{1.7}$$

Raising and lowering suffixes by means of the fundamental tensor corresponds in S_3 to taking the polar with respect to the fundamental quadric. Thus, if ξ^i is a point of S_3, then

$$\xi_i \equiv g_{ij}\xi^j$$

is its polar plane, of equation $\xi_i X^i = 0$ in current point-coordinates X^i. For this reason the operation of raising and lowering suffixes by means of the fundamental tensor will sometimes be referred to briefly as *polarizing*; and the operation of raising and lowering pairs of skew suffixes by means of the dualizors, as in (1.4), (1.5), (1.7), will be called *dualizing*. The two operations are geometrically equivalent, and also algebraically equivalent except perhaps for a multiplicative scalar, when the geometrical configuration represented by the operand is self-polar with respect to the fundamental quadric. Thus, for example, the line p^{ij} is self-polar when, and only when, there exists a scalar ρ such that

$$^\circ p_{ij} = \rho p_{ij}.$$

When $\rho = 1$, the operations are algebraically equivalent. It is to be observed that, in all cases, the two operations are commutative because—cf. (1.5)—

$$\tfrac{1}{2}\epsilon^{ijmn}g_{mnkl} \equiv \tfrac{1}{2}g^{ijmn}\epsilon_{mnkl}. \tag{1.8}$$

The word *polar* by itself will always refer to the polar with respect to the fundamental quadric.

The contravariant Riemann tensor R^{ijkl} and its dual ${}^\circ R_{ijkl}$ define the quadratic complex, of dual equations

$$R^{ijkl}\,{}^\circ p_{ij}\,{}^\circ p_{kl} = 0, \qquad {}^\circ R_{ijkl}\,p^{ij}p^{kl} = 0, \tag{1.9}$$

polar to the Riemann complex. ${}^\circ R_{ijkl}$ and R^{ijkl} are obtainable from one another by dualizing, and are respectively obtained from ${}^\circ R^{ijk}$

and R_{ijkl} by polarizing. The unsatisfactoriness of this notation from the geometrical point of view was commented upon in § 5 of the previous paper quoted above.

The Riemann complex is self-polar when

$$\kappa\,{}^\circ R^{ijkl} = R^{ijkl}. \tag{1.10}$$

It was shown in the previous paper that the scalar κ must have one of the values ± 1. When $\kappa = +1$, (1.10) is equivalent to the Einstein condition

$$R_{ij} = \tfrac{1}{4} g_{ij} R.$$

2. Ennuplet coordinates in S_3

Let $h_a^i \equiv (h_1^i, \ldots, h_4^i)$ be the components of an orthogonal ennuple at P in V_n. The letters $a, b, \ldots, g$ will always be used for ennuplet suffixes and $i, j, k, \ldots$ for proper tensor suffixes. We suppose that the h's satisfy the usual relations, and in particular

$$\sum_a h_a^i h_a^j = g^{ij}. \tag{2.1}$$

By taking a simple summation on the right-hand side, omitting the 'indicators' e_a $(= \pm 1)$ which correspond to the signature of the fundamental quadratic form (Eisenhart, **2**, ch. III), we are treating the h_a^i as though the fundamental form were positive-definite at (x^i). But the possibility of an indefinite metric need not be excluded, since it is of no consequence in the present theory if some of the h_a^i are imaginary.

Equation (2.1) may be written

$$g^{ab} h_a^i h_b^j = g^{ij}, \tag{2.2}$$

where g^{ab} is equal to the Kronecker symbol δ^{ab}. [This notation has a number of advantages: for example, the possibility of an indefinite metric may be provided for by taking the h's real and $g^{ab} = e_a \delta^{ab}$ (not summed) instead of assuming, as we are doing, that some of the h's may be imaginary.]

If $h \equiv \det|h_a^i|$, it follows from (2.2) that $h^2 = 1/g$, and we suppose that the signs of the h's are so chosen that

$$h > 0. \tag{2.3}$$

If g_{ab} is equal to the Kronecker symbol δ_{ab}, then also

$$g_{ab} h_i^a h_j^b = g_{ij},$$

where $[h_i^a]$ is the matrix reciprocal to $[h_a^i]$. Also

$$h_i^a = g^{ab} g_{ij} h_b^j.$$

Thus g^{ab} and g_{ab} may be used to raise and lower ennuplet suffixes.

The transformation $$X^a = h_i^a X^i$$

from tensor to ennuplet components of a contravariant vector corresponds in the projective space S_3 to a change of homogeneous coordinates, the vertices of the new tetrahedron of reference being the points whose coordinates in the X^i-system are $h_1^i,..., h_4^i$. The X^a are scalars for transformations of the coordinates x^i of V_4. Ennuplet components of tensors are distinguished from coordinate-components merely by having suffixes a, b,... instead of i, j,..., the same central letter being used in each case. The ennuplet components of the fundamental tensor itself are $g_{ij} h_a^i h_b^j$, and this is equal to g_{ab}. The equation of the fundamental quadric in S_3 in ennuplet coordinates is thus

$$g_{ab} X^a X^b = 0, \qquad \text{i.e.} \quad \sum_{a=1}^{4} (X^a)^2 = 0.$$

The new tetrahedron of reference is therefore self-polar with respect to the fundamental quadric.

The ennuplet components of the Riemann tensor are

$$R_{abcd} \equiv R_{ijkl} h_a^i h_b^j h_c^k h_d^l, \qquad R^{abcd} \equiv R^{ijkl} h_i^a h_j^b h_k^c h_l^d.$$

The ennuplet dualizors ϵ_{abcd} and ϵ^{abcd} both have components ± 1, 0, as follows from (2.3).

3. Five-dimensional representation

The representation of the lines of a projective 3-space by the points of a 4-quadric in a projective 5-space is too well known to need exposition (see, e.g., Sommerville, **8**, 343), but it is necessary to describe it briefly in order to explain the notation used in the present paper.

Let q^{ab} $(= -q^{ba})$ be the ennuplet components of a linear complex in S_3, its equation in current line-coordinates ${}^\circ p_{ab}$ being $q^{ab}\,{}^\circ p_{ab} = 0$, or dually, ${}^\circ q_{ab}\, p^{ab} = 0$. Write

$$\left.\begin{matrix} q^1 = q^{23}, & q^2 = q^{31}, & q^3 = q^{12} \\ q^4 = q^{14}, & q^5 = q^{24}, & q^6 = q^{34} \end{matrix}\right\}, \tag{3.1}$$

or, in one equation,

$$q^\alpha = \tfrac{1}{2}\gamma_{ab}^\alpha q^{ab} \quad (\alpha = 1, 2,..., 6). \tag{3.2}$$

The coefficients γ_{ab}^α are those of the linear transformation (3.1). Thus, for example,

$$\gamma_{23}^\alpha = -\gamma_{32}^\alpha = (1, 0, 0, 0, 0, 0).$$

Greek suffixes will always run from 1 to 6. The q^α may be interpreted as the homogeneous coordinates of a point in a projective space S_5. The transformation (3.2) may also be written

$$q^{ab} = \gamma^{ab}_\alpha q^\alpha, \tag{3.3}$$

where $\gamma^{23}_\alpha = -\gamma^{32}_\alpha = (1, 0, 0, 0, 0, 0)$, etc.

As it stands at present *the transformation amounts merely to the replacement of pairs of skew suffixes by single suffixes according to the scheme*

$$\left.\begin{array}{rcccccc} ab = -ba = & 23 & 31 & 12 & 14 & 24 & 34 \\ \alpha = & 1 & 2 & 3 & 4 & 5 & 6 \end{array}\right\}. \tag{3.4}$$

By (3.2) and (3.3) it quickly follows that

$$\tfrac{1}{2}\gamma^\alpha_{ab}\gamma^{ab}_\beta = \delta^\alpha_\beta, \tag{3.5}$$

and that

$$\gamma^{ab}_\alpha \gamma^\alpha_{cd} = \delta^{ab}_{cd}, \tag{3.6}$$

where δ^{ab}_{cd} is the generalized Kronecker symbol $\delta^a_c \delta^b_d - \delta^a_d \delta^b_c$. Write

$$\epsilon_{\alpha\beta} \equiv \tfrac{1}{4}\epsilon_{abcd}\gamma^{ab}_\alpha \gamma^{cd}_\beta, \qquad \epsilon^{\alpha\beta} \equiv \tfrac{1}{4}\epsilon^{abcd}\gamma^\alpha_{ab}\gamma^\beta_{cd}. \tag{3.7}$$

Then $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$ are symmetric in α, β, and

$$\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha_\beta. \tag{3.8}$$

Also

$$\epsilon_{abcd} = \epsilon_{\alpha\beta}\gamma^\alpha_{ab}\gamma^\beta_{cd}, \qquad \epsilon^{abcd} = \epsilon^{\alpha\beta}\gamma^{ab}_\alpha \gamma^{cd}_\beta. \tag{3.9}$$

Equations (3.7) are a disguised form of the statement—cf. (3.4)—that

$$\begin{aligned} \epsilon_{14} = \epsilon_{41} = \epsilon_{2314} = 1, \\ \epsilon_{24} = \epsilon_{42} = \epsilon_{3124} = 1, \\ \epsilon_{34} = \epsilon_{43} = \epsilon_{1234} = 1, \end{aligned}$$

with $\epsilon_{\alpha\beta} = 0$ otherwise, and similarly for $\epsilon^{\alpha\beta}$. Thus

$$\epsilon_{\alpha\beta} = \begin{bmatrix} O & I \\ I & O \end{bmatrix} = \epsilon^{\alpha\beta} \quad (\alpha \text{ row}, \beta \text{ column}), \tag{3.10}$$

I and O being the unit and null 3-by-3 matrices respectively. If

$$g_{\alpha\beta} \equiv \tfrac{1}{4}g_{abcd}\gamma^{ab}_\alpha \gamma^{cd}_\beta, \qquad g^{\alpha\beta} \equiv \tfrac{1}{4}g^{abcd}\gamma^\alpha_{ab}\gamma^\beta_{cd}, \tag{3.11}$$

where, as usual,

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc},$$

then

$$g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta. \tag{3.12}$$

Because g_{ab}, g^{ab} are both equal to the Kronecker delta, it follows that

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \qquad g^{\alpha\beta} = \delta^{\alpha\beta}, \qquad \text{i.e. } g_{\alpha\beta} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = g^{\alpha\beta}. \tag{3.13}$$

The formalism may seem a trifle elaborate for so simple a transformation as that implied by (3.4). The reason for its introduction is twofold: it makes possible the use of Greek suffixes as applying to S_5, and also frees the theory from the restriction to the particular coordinate-systems implied by the choice of the ennuple h^i_a in S_3 and by the transformation (3.1). Thus, if we write

$$\gamma^\alpha_{ij} = \gamma^\alpha_{ab} h^a_i h^b_j,$$

then, for fixed α, γ^α_{ij} is a covariant six-vector of V_4 which defines a linear complex in S_3, and (3.2) may be written

$$q^\alpha = \tfrac{1}{2}\gamma^\alpha_{ij} q^{ij}. \tag{3.14}$$

Moreover, in S_5 the coordinates q^α may be subjected to linear transformations, say

$$q'^\alpha = l^\alpha_\beta q^\beta,$$

and (3.14) then becomes

$$q'^\alpha = \gamma'^\alpha_{ij} q^{ij},$$

where

$$\gamma'^\alpha_{ij} = l^\alpha_\beta \gamma^\beta_{ij}.$$

In other words, the coordinates of S_3 and S_5 may be transformed independently of one another if γ^α_{ij} is treated as a tensor of the type indicated by its suffixes. In all proper tensor formulae, such as (3.5), (3.6), (3.7), and (3.8), ennuplet suffixes may be replaced by proper tensor suffixes, or may be taken to refer to an ennuple other than h^i_a, and the Greek suffixes may be taken to refer to an arbitrary coordinate-system in S_5. For general coordinate-systems the numerical values of $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $g_{\alpha\beta}$, $g^{\alpha\beta}$ given by (3.10) and (3.13) do not hold. When they do hold, that is, when the transformation from S_3 to S_5 is given by (3.1), the coordinate-systems q^α and X^a in S_3 and S_5 respectively will be called *basic*, and the orthogonal ennuple h^i_a will be called the *basis* of the transformation.

To pass from a tensor equation in V_4 or S_3 to the corresponding equation for S_5, it is therefore necessary only to replace pairs of skew suffixes (tensor or ennuplet) by single Greek suffixes, and, when necessary, to calculate factors of proportionality by reference to the basic coordinate-system.

Now the coordinates p^{ij} of a line in S_3 satisfy the identity

$$p^{23}p^{14}+p^{31}p^{24}+p^{12}p^{34} = 0,$$

which, multiplied by $2\sqrt{g}$, may be written

$$\tfrac{1}{4}\epsilon_{ijkl}p^{ij}p^{kl} = 0, \quad \text{or, dually,} \quad \tfrac{1}{4}\epsilon^{ijkl}\,{}^\circ p_{ij}\,{}^\circ p_{kl} = 0. \tag{3.15}$$

The corresponding equations for S_5 are

$$\epsilon_{\alpha\beta}p^{\alpha}p^{\beta} = 0 \tag{3.16}$$

and

$$\epsilon^{\alpha\beta\circ}p_{\alpha}{}^{\circ}p_{\beta} = 0, \tag{3.17}$$

where

$$^{\circ}p_{\alpha} = \epsilon_{\alpha\beta}p^{\beta}. \tag{3.18}$$

Of these (3.16) is the point-equation in current coordinates p^{α} of the non-degenerate 4-quadric in S_5 the points of which correspond to the lines of S_3. Because of (3.8) it is evident that (3.17) is the tangential equation of the same 4-quadric in current hyperplane-coordinates $^{\circ}p_{\alpha}$. For any point p^{α}, (3.18) defines its polar 4-plane with respect to the 4-quadric $\epsilon_{\alpha\beta}$, which will be called the *ϵ-quadric.*

The lines of the special quadratic complex

$$g_{ijkl}p^{ij}p^{kl} = 0 \tag{3.19}$$

in S_3 correspond in S_5 to the points of the 3-space of intersection of the non-degenerate 4-quadric

$$g_{\alpha\beta}p^{\alpha}p^{\beta} = 0 \tag{3.20}$$

with the ϵ-quadric. We call this the *g-quadric*, the term 'fundamental' being inappropriate, even though (3.20) is obtained from the line-equation (3.19) of the fundamental quadric in S_3, because of the even more fundamental nature of the ϵ-quadric in S_5. By (3.12) the tangential equation of the g-quadric is

$$g^{\alpha\beta\circ}p_{\alpha}{}^{\circ}p_{\beta} = 0.$$

By (1.5),

$$\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}g_{\gamma\delta} = g^{\alpha\beta}. \tag{3.21}$$

The left-hand side represents the polar of the g-quadric with respect to the ϵ-quadric, and the right-hand side is the tangential form of the g-quadric. Hence *the g-quadric is self-polar with respect to the ϵ-quadric.* Similarly, since

$$\tfrac{1}{4}g^{ijmn}g^{klpq}\epsilon_{mnpq} = \epsilon^{ijkl},$$

(this being merely the statement that the ϵ-tensors are obtainable from one another by polarizing), we have, in S_5,

$$g^{\alpha\gamma}g^{\beta\delta}\epsilon_{\gamma\delta} = \epsilon^{\alpha\beta}. \tag{3.22}$$

Therefore *the ϵ-quadric is self-polar with respect to the g-quadric.* Equations (3.21) and (3.22) are both deducible from

$$\epsilon^{\alpha\gamma}g_{\gamma\beta} = g^{\alpha\gamma}\epsilon_{\gamma\beta}$$

—cf. (1.8)—which states that the operations of taking the polars with respect to the two quadrics are commutative.

The polars of any geometrical configuration in S_5 with respect to the quadrics will be called its *ϵ-polar* and *g-polar* respectively.

The lines of the Riemann complex

$$R_{ijkl}p^{ij}p^{kl}=0$$

in S_3 correspond in S_5 to the points common to the 4-quadric

$$R_{\alpha\beta}p^{\alpha}p^{\beta}=0 \tag{3.23}$$

and the ϵ-quadric. Here

$$R_{\alpha\beta}=R_{\beta\alpha}=\tfrac{1}{4}\gamma^{ij}_{\alpha}\gamma^{kl}_{\beta}R_{ijkl}. \tag{3.24}$$

The dual equation of the complex in S_3 is

$$^{\circ}R^{ijkl}\,{}^{\circ}p_{ij}\,{}^{\circ}p_{kl}=0, \tag{3.25}$$

and the corresponding equation for S_5 is

$$^{\circ}R^{\alpha\beta}\,{}^{\circ}p_{\alpha}\,{}^{\circ}p_{\beta}=0, \tag{3.26}$$

where

$$^{\circ}R^{\alpha\beta}=\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}R_{\gamma\delta}=\tfrac{1}{4}\gamma^{\alpha}_{ij}\gamma^{\beta}_{kl}\,{}^{\circ}R^{ijkl}. \tag{3.27}$$

In the basic coordinate-system (3.24) is

$$R_{\alpha\beta}=\begin{bmatrix} R_{2323} & R_{2331} & . & . & . & R_{2334} \\ R_{3123} & R_{3131} & . & . & . & R_{3134} \\ . & . & . & . & . & . \\ R_{3423} & R_{3431} & . & . & . & R_{3434} \end{bmatrix}, \tag{3.28}$$

the (ennuplet) suffix-pairs on the right running according to the scheme (3.4). The symmetric matrix (3.28) may be symbolized by

$$R_{\alpha\beta}=\begin{bmatrix} P & S \\ S' & Q \end{bmatrix}, \tag{3.29}$$

where, of the 3-by-3 matrices P, Q, S, S', P and Q are symmetric and S' is the transpose of S.

By (3.27) $^{\circ}R^{\alpha\beta}$ is the 4-quadric (envelope) in S_5 polar to the 4-quadric $R_{\alpha\beta}$ with respect to the ϵ-quadric. The point-equation of the 4-quadric $^{\circ}R^{\alpha\beta}$ is

$$^{\circ}R^{*}_{\alpha\beta}p^{\alpha}p^{\beta}=0,$$

where the coefficients $^{\circ}R^{*}_{\alpha\beta}$ are such that

$$^{\circ}R^{\alpha\gamma}\,{}^{\circ}R^{*}_{\gamma\beta}=\kappa\delta^{\alpha}_{\beta}, \tag{3.30}$$

κ being a scalar (or zero if the 4-quadric is degenerate). In general the tensor $^{\circ}R^{*}_{\alpha\beta}$ is not the same as the original tensor $R_{\alpha\beta}$, nor even a scalar multiple of it: it is so only when the quadric $R_{\alpha\beta}$ is self-polar with respect to the ϵ-quadric, and in that case, by (3.30),

$$^{\circ}R^{\alpha\gamma}R_{\gamma\beta}=\rho\delta^{\alpha}_{\beta}\quad (\rho \text{ scalar}). \tag{3.31}$$

In S_3 this becomes $\qquad {}^\circ R^{ijmn} R_{mnkl} = \rho \delta^{ij}_{kl},$ **(3.32)**

which, as will be seen in § 5, equation (5.5), is the condition that the Riemann complex should be special, that is, consist of the tangents to a quadric, or, degenerately, of the tangents to a cone or of the lines meeting a proper or degenerate conic.

Similarly, the tangential equation of the quadric $R_{\alpha\beta}$ is

$$R^{*\alpha\beta} {}^\circ p_\alpha {}^\circ p_\beta = 0,$$

where $R^{*\alpha\beta}$ is such that $\qquad R^{*\alpha\gamma} R_{\gamma\beta} = \mu \delta^\alpha_\beta,$

and $R^{*\alpha\beta}$ is the same as ${}^\circ R^{\alpha\beta}$, or is a scalar multiple of it, only when the complex in S_3 is special.

As it is desirable as far as possible to avoid a confusion of notations, no more use will be made of the tensors ${}^\circ R^*_{\alpha\beta}$ and $R^{*\alpha\beta}$, which correspond in S_3 to the quadratic complex *conjugate* to the Riemann complex (Salmon, **7**, 51).

In addition to the quadric $R_{\alpha\beta}$ and its ϵ-polar ${}^\circ R^{\alpha\beta}$, we have, in S_5, the 4-quadric (envelope) defined by

$$R^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} R_{\gamma\delta}$$

which corresponds to the polar complex R^{ijkl} in S_3, and the 4-quadric (locus) defined by

$$ {}^\circ R_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} {}^\circ R^{\gamma\delta},$$

which corresponds in S_3 to the dual ${}^\circ R_{ijkl}$ of the polar complex R^{ijkl}. The former is the g-polar of the original 4-quadric $R_{\alpha\beta}$. The latter is the g-polar of the 4-quadric ${}^\circ R^{\alpha\beta}$, which is itself the ϵ-polar of $R_{\alpha\beta}$. The envelope $R^{\alpha\beta}$ has a point-equation $R^*_{\alpha\beta} p^\alpha p^\beta = 0$, and the locus ${}^\circ R_{\alpha\beta}$ a tangential form ${}^\circ R^{*\alpha\beta}$, but, once again, we omit these from consideration.

To sum up: In S_5 we have, fundamentally, three 4-quadrics $\epsilon_{\alpha\beta}$, $g_{\alpha\beta}$, $R_{\alpha\beta}$, the first being the one whose points correspond to the lines of S_3. The quadrics $\epsilon_{\alpha\beta}$ and $g_{\alpha\beta}$ are non-degenerate, and each is self-polar with respect to the other. $R_{\alpha\beta}$ may or may not be degenerate according to the nature of the quadratic complex in S_3, the lines of which are represented in S_5 by the intersection of $\epsilon_{\alpha\beta}$ and $R_{\alpha\beta}$. From the quadric $R_{\alpha\beta}$ we get, taking polars,

(i) its ϵ-polar (envelope) ${}^\circ R^{\alpha\beta}$;

(ii) its g-polar (envelope) $R^{\alpha\beta}$;

(iii) the g-polar ${}^\circ R_{\alpha\beta}$ of ${}^\circ R^{\alpha\beta}$, which is the same locus as the ϵ-polar of $R^{\alpha\beta}$, because the operations of taking the ϵ- and g-polars are commutative.

We shall not need to consider the polars of the ϵ- and g-quadrics with respect to the R-quadrics.

It is perhaps not surprising that such a wealth of geometrical material leads to a fairly extensive theory, to which the present paper is intended to serve as an introduction.

4. The Segre characteristics of the Riemann tensor at a point of V_4

Let q^α be a point of S_5. Its polar 4-planes with respect to the R- and ϵ-quadrics have coordinates $R_{\alpha\beta}q^\beta$, $\epsilon_{\alpha\beta}q^\beta$, and are the same if

$$R_{\alpha\beta}q^\beta = \lambda\epsilon_{\alpha\beta}q^\beta,$$

that is, if
$$(R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta})q^\beta = 0, \tag{4.1}$$

λ being a (scalar) root of the equation

$$\det|R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta}| = 0 \tag{4.2}$$

of the sixth degree. When the roots are all distinct, there is one point q^α corresponding to each root, the six points so obtained being the vertices of the common self-polar simplex of the two quadrics. In S_3 the equation corresponding to (4.1) is

$$(R_{ijkl}-\lambda\epsilon_{ijkl})q^{kl} = 0, \tag{4.3}$$

and q^{ij} is a *principal linear complex* of the quadratic complex R_{ijkl}. In V_4 each q^{ij} is a contravariant six-vector.

Whether the roots of (4.1) are distinct or not, the quadratic complex R_{ijkl} may be classified by means of the Segre characteristic of the matrix pencil $[R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta}]$ formed from the exponents of the elementary divisors. This provides the usual classification of the complex as one of 49 different species (Jessop, **3**, 230–2; or Zindler, **10**, 1128–31), or as one of 8 degenerate forms (Zindler, ibid., 1133). In my previous paper (Ruse, **6**, § 4) it was seen that there is no corresponding theory for a V_3.

This classification takes no account, however, of the relation of the quadratic complex to the fundamental quadric. This relation may be similarly symbolized by the Segre characteristic of the pencil $[R_{\alpha\beta}-\mu g_{\alpha\beta}]$. This corresponds in a general sense to the classification of a V_3 by means of the relation between the Riemann conic, or equivalently of the Ricci conic, and the fundamental conic (Ruse, **6**). This simultaneous use of the Segre characteristics of $[R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta}]$ and $[R_{\alpha\beta}-\mu g_{\alpha\beta}]$, which will be called the ϵ- and *g-discriminants* respec-

tively, obviously gives a more detailed classification of the Riemannian V_4 than is afforded by the roots of the equation

$$\det|R_{ij}-\rho g_{ij}| = 0 \tag{4.4}$$

appearing in the theory of Ricci principal directions.

The two characteristics will be called the ϵ- and *g-characteristics* respectively, and the roots of the discriminant equations the ϵ- and *g-roots*.

Although the characteristics may be used for a fairly detailed description of the nature of a V_4 at a point, it should be noted that they do not by themselves give full information about the Riemann tensor at that point: they take into account only *equalities* among the ϵ- and g-roots, and the distribution of those roots among the minors of the corresponding matrices. Yet the roots may be connected by relations other than that of simple equality, as, for example, in the case of the harmonic complex (see § 5 below), which, though a complex of a particular sort, may have any one of a number of ϵ- and g-characteristics, including [111111] (Jessop, **3**, 359).

That both characteristics enter naturally into a discussion of the properties of the Riemann tensor in a V_4 has not been fully recognized, or at any rate not emphasized, by previous writers. Struik (**9**) and Lamson (**4**) refer explicitly only to the ϵ-characteristic, though Lamson, who confines his discussion to the Einstein V_4 ($R_{ij} = kg_{ij}$), takes full account of the geometrical relationship of the Riemann complex to the fundamental quadric. Churchill (**1**, § 11), on the other hand, takes explicit notice only of the g-discriminant.

5. Special cases

Consider the case when the V_4 is of constant curvature. Then at all points

$$R_{ijkl} = K(g_{ik}g_{jl}-g_{il}g_{jk}).$$

Hence—see (1.2)—the Riemann complex coincides with the special complex of tangents to the fundamental quadric, and is therefore (Jessop, **3**, 211) of ϵ-characteristic [(111)(111)]. Now every special quadratic complex has the same ϵ-characteristic, so that a space for which

$$R_{ijkl} = \pm(a_{ik}a_{jl}-a_{il}a_{jk}), \tag{5.1}$$

where a_{ij} is any symmetric tensor of non-zero determinant, is also of ϵ-characteristic [(111)(111)]. But any V_4 immersible in a flat space

of five dimensions is of this type provided that its second fundamental tensor b_{ij} is of non-zero determinant, because, for such a V_4,

$$R_{ijkl} = \pm(b_{ik}b_{jl} - b_{il}b_{jk}) \tag{5.2}$$

(Eisenhart, **2**, 197). Thus a classification of Riemannian spaces by means of the ϵ-characteristic alone, without reference to the relation of the complex to the fundamental quadric, would treat as one class all hypersurfaces of a flat 5-dimensional space. This adds point to the remarks made in the last section about the significance of the g-characteristic.

Stating the above result as a theorem, we therefore have:

Every Riemannian V_4 which is a hypersurface of a flat V_5 is of ϵ-characteristic $[(111)(111)]$ *at all non-singular points, provided that its second fundamental form is of non-zero determinant.*

If it is everywhere of g-characteristic $[(111111)]$, that is, if R_{ijkl} is a constant multiple of g_{ijkl}, then it is a space of constant curvature.

When the determinant of a_{ij} in (5.1) is zero, the Riemann complex is even more specialized. If the matrix $[a_{ij}]$ is of rank 3, the complex consists of all lines touching the cone $a_{ij}X^iX^j = 0$ and so is of ϵ-characteristic $[(222)]$ (Jessop, **3**, 226; an example of this type of V_4 is given below). If $[a_{ij}]$ is of rank 2, the quadric a_{ij} is a pair of planes, and the Riemann complex degenerates into the square of the special linear complex having the line of intersection of the two planes as directrix; that being so, the ϵ-characteristic is now $[(21111)]$ (Zindler, **10**, 1133; an example of this type of V_4 is also given below). The case when $[a_{ij}]$ is of rank 1 does not arise, because, if it were of rank 1, all components of the Riemann tensor would be zero.

Although it is not proposed to find analytical conditions for all the special cases considered, it may be of interest to obtain by a geometrical argument a necessary condition for R_{ijkl} to be of the form (5.1). The complex consists of tangents to the quadric a_{ij}, which we assume for the moment to be non-degenerate. Hence the polar linear complex $R_{ijkl}p^{kl}$ of any line p^{ij} is special, its directrix being the polar of p^{ij} with respect to the quadric. So $R_{ijkl}p^{kl}$ satisfies the conditions for a line, whence

$$\tfrac{1}{4}\epsilon^{ijkl}R_{ijmn}R_{klrs}p^{mn}p^{rs} = 0. \tag{5.3}$$

This is true for any line p^{ij}, and so must be a consequence of the identity

$$\tfrac{1}{4}\epsilon_{mnrs}p^{mn}p^{rs} = 0$$

satisfied by the coordinates of a line. Therefore

$$\tfrac{1}{4}\epsilon^{ijkl}R_{ijmn}R_{klrs} = \tfrac{1}{4}\mu\epsilon_{mnrs},$$

where μ is a scalar. Multiplying by ϵ^{mnpq} and summing for m, n, we get

$$°R^{klpq}R_{klrs} = \tfrac{1}{2}\mu\delta^{pq}_{rs}, \tag{5.4}$$

and hence, contracting r with p and q with s,

$$(°RR) = 6\mu,$$

where
$$(°RR) \equiv °R^{klpq}R_{klpq}.$$

Substituting in (5.4), we deduce that, *if the Riemann complex is of ϵ-characteristic* $[(111)(111)]$, *the Riemann tensor being of the form* (5.1) *with* $\det|a_{ij}| \neq 0$, *then*

$$°R^{ijmn}R_{klmn} = \tfrac{1}{12}(°RR)\delta^{ij}_{kl}. \tag{5.5}$$

This result may also be established purely analytically. Reference to it was made in § 3 (3.32).

Conversely, it can be shown that, if R_{ijkl} satisfies (5.5), and if $(°RR) \neq 0$, then the Riemann complex consists of tangents to a quadric. If, however, it satisfies (5.5) with $(°RR) = 0$, then—cf. Ruse, **5**, 450 (4.4)—the complex degenerates into the set of lines meeting a conic, or dually, touching a cone (case $[(222)]$); or it degenerates still further into the set of lines meeting two different intersecting lines (case $[(2211)]$) or meeting two coincident lines (case $[(21111)]$).

An example of the case of ϵ-characteristic $[(222)]$ is provided by the V_4

$$ds^2 = t^2(dx^2+dy^2+dz^2)+dt^2 \quad (x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = t),$$

for which
$$R_{2323} = -t^2 = R_{3131} = R_{1212},$$

the remaining effectively different components being all zero. In this case the equation of the Riemann complex at the point (x, y, z, t) of V_4 is

$$-t^2\{(p^{23})^2+(p^{31})^2+(p^{12})^2\} = 0,$$

which is the complex of lines touching the cone

$$X^2+Y^2+Z^2 = 0$$

in S_3. The Ricci quadric is degenerate and coincides with this cone, the only non-zero components of the Ricci tensor being

$$R_{11} = -2 = R_{22} = R_{33}.$$

The g-characteristic of this space is easily seen to be $[(111)(111)]$.

For the V_4 defined by

$$ds^2 = t^2(dx^2+dy^2)+dz^2+dt^2$$

all the components of the Riemann tensor are zero except

$$R_{1212} = -t^2 \quad (= -R_{2112}, \text{ etc.}).$$

When $t \neq 0$, the ϵ-characteristic is $[(21111)]$. The equation of the Riemann complex is

$$-t^2(p^{12})^2 = 0,$$

so the complex consists of the special linear complex $p^{12} = 0$ taken twice. The g-characteristic is $[(11111)1]$. It is perhaps rather surprising that a Riemannian V_4 can be as specialized as this without being actually flat.

Reference was made in my previous paper (Ruse, **6**, (6.13) *et seq.*) to the fact that any quadratic complex which is self-polar of the second kind with respect to the fundamental quadric is a harmonic complex. In general, if the Riemann tensor is of the form

$$R_{ijkl} = a_{ik}b_{jl}+a_{jl}b_{ik}-a_{il}b_{jk}-a_{jk}b_{il}, \tag{5.6}$$

where a_{ij}, b_{ij} are symmetric, then the Riemann complex consists of the lines which cut the quadrics a_{ij}, b_{ij} in harmonically conjugate points, and is of the type known as harmonic.

The Riemann complex for every V_4 conformal to a flat space is harmonic, because, for such a V_4,

$$R_{ijkl} = -(g_{ik}d_{jl}+g_{jl}d_{ik}-g_{il}d_{jk}-d_{jk}g_{il}), \tag{5.7}$$

where $d_{ij} \equiv \frac{1}{2}R_{ij}-\frac{1}{12}g_{ij}R$ (Eisenhart, **2**, 217–18). A harmonic complex is in general of ϵ-characteristic $[111111]$, but may have various other ϵ-characteristics according to the relations between the quadrics a_{ij}, b_{ij} which determine it (Jessop, **3**, 359). For example, the Riemann complex for the Schwarzschild space-time

$$ds^2 = -(1-2m/r)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta\, d\phi^2+(1-2m/r)\,dt^2 \tag{5.8}$$

is of ϵ-characteristic $[(11)(11)11]$ (Lamson, **4**, 722), and is a special case of the harmonic complex in which the defining quadrics a_{ij}, b_{ij} of (5.6) have four common generators forming a skew quadrilateral. The singular surface (Ruse, **6** (5.15)) consists of two quadrics meeting in the same skew quadrilateral, their equations being

$$\alpha_{ij}X^iX^j \equiv -2\mu^{-1}X^2+r^2Y^2+r^2\sin^2\theta Z^2+2\mu T^2 = 0,$$
$$\beta_{ij}X^iX^j \equiv -\mu^{-1}X^2+2r^2Y^2+2r^2\sin^2\theta Z^2+\mu T^2 = 0,$$

where $\mu = 1-2m/r$, while the fundamental quadric is

$$g_{ij}X^iX^j \equiv -\mu^{-1}X^2-r^2Y^2-r^2\sin^2\theta Z^2+\mu T^2 = 0.$$

The quadrics a_{ij}, b_{ij} of (5.6) are in this case given by

$$a_{ij} = \left(\frac{m}{3r^3}\right)^{\frac{1}{2}}(\alpha_{ij}+\omega\beta_{ij}), \qquad b_{ij} = -\left(\frac{m}{3r^3}\right)^{\frac{1}{2}}(\alpha_{ij}+\omega^2\beta_{ij}),$$

where ω is a complex cube root of unity. Also

$$g_{ij} = \alpha_{ij}-\beta_{ij},$$

and the quadrics a_{ij}, b_{ij}, α_{ij}, β_{ij}, g_{ij} therefore all belong to the same pencil and intersect in the same skew quadrilateral. Moreover, it is easy to see that the quadrics α_{ij}, β_{ij} are polar to one another with respect to the fundamental quadric, and hence also that a_{ij}, b_{ij} are polar to one another.

The g-characteristic of the Schwarzschild space-time (5.8) is [(1111)(11)]. In general, however, the g-characteristic of a harmonic Riemann complex, whether of the form (5.6) or of the more specialized form (5.7), is [111111]. When the ϵ- and g-characteristics of (5.7) are both [111111], there are no equalities among the ϵ-roots $\lambda_1,\ldots, \lambda_6$ or among the g-roots $\mu_1,\ldots, \mu_6$, but they are separable into pairs such that

$$\lambda_1+\lambda_4 = 0 = \lambda_2+\lambda_5 = \lambda_3+\lambda_6,$$

$$\mu_1+\mu_4 = 0 = \mu_2+\mu_5 = \mu_3+\mu_6,$$

(cf. Jessop, **3**, 134).

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ON LATTICE POINTS IN A CYLINDER

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DENOTE by x_1, x_2, x_3 rectangular coordinates in three-dimensional space, and by K a convex body with the origin $O = (0, 0, 0)$ as its centre. A lattice Λ,

$$x_h = \sum_{k=1}^{3} \alpha_{hk} u_k \quad (h = 1, 2, 3;\ u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots),$$

say of determinant

$$d(\Lambda) = \left| |\alpha_{hk}|_{h,k=1,2,3} \right|,$$

is called K-*admissible* if O is its only point which is an *inner* point of K. Let $\Delta(K)$ be the lower bound of $d(\Lambda)$ extended over all K-admissible lattices. Then $\Delta(K) > 0$, and there is at least one *critical lattice*, i.e., a K-admissible lattice Λ such that $d(\Lambda) = \Delta(K)$.*

Minkowski's theorem on convex bodies may be expressed as

$$\Delta(K) \geqslant \tfrac{1}{8} V,$$

where V is the volume of K. In general, the sign '$>$' holds in this inequality, and so the problem arises of finding the exact value of $\Delta(K)$. Minkowski himself solved this problem for the cube, the octahedron, and the sphere. In this note I solve it for the cylinder

$$K: \quad x_1^2 + x_2^2 \leqslant 1, \quad -1 \leqslant x_3 \leqslant +1,$$

by proving that†

$$\Delta(K) = \tfrac{1}{2}\sqrt{3}, \tag{1}$$

a result which surprisingly has escaped notice.

That $\Delta(K) \leqslant \frac{1}{2}\sqrt{3}$ is nearly trivial, because the following lattices of determinant $\frac{1}{2}\sqrt{3}$ are evidently K-admissible:

(i) The lattices Λ_1 derived from the particular lattice

$$x_1 = u_1 + \tfrac{1}{2}u_2 + \alpha u_3, \quad x_2 = \tfrac{1}{2}\sqrt{3}\, u_2 + \beta u_3, \quad x_3 = u_3 \quad (\alpha, \beta \text{ arbitrary})$$

by any rotation about the x_3-axis;

* For the two-dimensional case of these rather obvious statements see my note, *J. of London Math. Soc.* 17 (1942), 130–3.

† The same proof shows that $\Delta(K) = \frac{1}{2}\sqrt{3}$, where K is the n-dimensional convex body $x_1^2 + x_2^2 \leqslant 1$, $|x_3| \leqslant 1, \ldots, |x_n| \leqslant 1$ $(n \geqslant 2)$.

(ii) The lattices Λ_2 derived from the particular lattice

$$x_1 = u_1+\tfrac{1}{2}u_2, \quad x_2 = \tfrac{1}{2}\surd 3\, u_2, \quad x_3 = \alpha u_1+\beta u_2+u_3 \quad (\alpha, \beta \text{ arbitrary})$$

by any rotation about the x_3-axis.*

It suffices therefore to show that

$$\Delta(K) \geqslant \tfrac{1}{2}\surd 3, \tag{2}$$

in order to prove the assertion (1).

I use the following lemmas.

LEMMA 1. *Let π be a plane convex polygon of area A with angles not greater than* 120°, *and let $C_1, C_2,..., C_s$ be non-overlapping circles of radius r contained in π. Then*

$$s \leqslant \frac{A}{r^2\surd 12}.\dagger$$

LEMMA 2. *Let n be a positive integer, and let W be the cube*

$$|x_1| \leqslant n, \qquad |x_2| \leqslant n, \qquad |x_3| \leqslant n.$$

Let further $Z_1, Z_2,..., Z_t$ be non-overlapping circular cylinders of radius $\frac{1}{2}$ *and height* 1, *all contained in W with their axes parallel to the x_3-axis. Then*

$$t \leqslant \frac{16}{\surd 3}n^3.$$

Proof. Denote by x any number in the interval $-n \leqslant x \leqslant n$. The plane $x_3 = x$ intersects the cylinders $Z_1, Z_2,..., Z_t$ in a certain point set $J(x)$, say of area $Q(x)$. Then the integral $\int\limits_{-n}^{+n} Q(x)\,dx$ equals the total volume of the cylinders $Z_1, Z_2,..., Z_t$, and so

$$\int\limits_{-n}^{+n} Q(x)\,dx = \tfrac{1}{4}\pi t. \tag{3}$$

Now there are at most $2t$ different values of x for which the plane $x_3 = x$ contains either the base or the top of one of these cylinders; let x be different from these exceptional values. Then $J(x)$ consists of a finite number, say s, of circles of radius $\frac{1}{2}$; no two of these circles overlap, and all lie inside the square

$$|x_1| \leqslant n, \qquad |x_2| \leqslant n, \qquad x_3 = x$$

* The lattices Λ_1 and Λ_2 are the only critical lattices of K, as can be proved. One can further show that, if H is any convex body symmetrical in O which is contained in, but different from K, then $\Delta(H) < \Delta(K)$.

† For a proof see the note 'On the densest packing of circles' by B. Segre and myself, *American Math. Monthly*, 51 (1944), 261–70.

of area $A = 4n^2$. Hence, by Lemma 1,

$$s \leqslant \frac{4n^2}{(\frac{1}{2})^2\sqrt{12}} = \frac{8}{\sqrt{3}}n^2.$$

Then
$$Q(x) = \tfrac{1}{4}\pi s \leqslant \frac{2\pi}{\sqrt{3}}n^2,$$

and so, by (3),

$$\tfrac{1}{4}\pi t = \int\limits_{-n}^{+n} Q(x)\,dx \leqslant \int\limits_{-n}^{+n} \frac{2\pi}{\sqrt{3}}n^2\,dx = \frac{4\pi}{\sqrt{3}}n^3,$$

whence
$$t \leqslant \frac{16}{\sqrt{3}}n^3.$$

Proof of (2). Put

$$F(x_1, x_2, x_3) = \max(|\sqrt{(x_1^2+x_2^2)}|,\ |x_3|),$$

so that the cylinder K consists of all points satisfying $F(x_1, x_2, x_3) \leqslant 1$ Denote by Λ any K-admissible lattice. Then at every point $X = (x_1^0, x_2^0, x_3^0)$ of Λ place a cylinder

$Z(X)$:
$$F(x_1-x_1^0, x_2-x_2^0, x_3-x_3^0) \leqslant \tfrac{1}{2}$$

of half the linear dimensions of K, and with its centre at R and axis parallel to the x_3-axis. Since Λ is K-admissible and since K is convex, no two of these cylinders overlap.*

Let n now be a large positive integer. Since every lattice parallelepiped is of volume $d(\Lambda)$, the cube

$$|x_1| \leqslant n-\tfrac{1}{2}, \qquad |x_2| \leqslant n-\tfrac{1}{2}, \qquad |x_3| \leqslant n-\tfrac{1}{2}$$

contains
$$\frac{8n^3}{d(\Lambda)} + O(n^2)$$

points X of Λ; at least as many cylinders $Z(X)$ lie therefore in the cube

$$|x_1| \leqslant n, \qquad |x_2| \leqslant n, \qquad |x_3| \leqslant n.$$

Thus, by Lemma 2,

$$\frac{8n^3}{d(\Lambda)} + O(n^2) \leqslant \frac{16}{\sqrt{3}}n^3,$$

whence

$$d(\Lambda) \geqslant \tfrac{1}{2}\sqrt{3}-o(1), \quad \text{i.e.} \quad d(\Lambda) \geqslant \tfrac{1}{2}\sqrt{3}, \qquad \Delta(K) \geqslant \tfrac{1}{2}\sqrt{3},$$

as asserted.

* Minkowski, *Geometrie der Zahlen*, 74.

ON AN INTEGRAL EQUATION ARISING IN THE THEORY OF DIFFRACTION

By E. T. COPSON (*Dundee*)

[Received 30 September 1944]

1. Introduction

THE problem of the diffraction of plane sound waves of small amplitude or plane electromagnetic waves by a perfectly reflecting half-plane was first solved rigorously by Sommerfeld* in 1896 when he introduced many-valued solutions of the equation of wave-motions having the geometrical shadow as branch membrane. Sommerfeld's results were subsequently obtained by H. M. Macdonald† who expanded the wave-function as a series of Bessel functions and by Lamb‡ (for the case of normal incidence) who used parabolic coordinates. Although the subject already has a large literature,§ an interesting new contribution has recently been made by W. Magnus,|| who has shown that the problem can be reduced to the solution of a singular integral equation.

Magnus arrives at his integral equation by the following physical argument. If plane polarized electromagnetic waves with electric force $E_0 e^{i\omega t}$ in the direction of the axis of z are incident on the perfectly conducting half-plane $y = 0$, $x > 0$, then, as Poincaré pointed out many years ago, an alternating current-sheet flows in the conductor in the direction of the axis of z and produces the scattered field $E_1 e^{i\omega t}$ which is also polarized parallel to Oz. This field is given by

$$E_1 = \int\limits_0^\infty f(\xi)H_0^{(2)}[k\surd\{(x-\xi)^2+y^2\}]\,d\xi,$$

where the function $f(\xi)$, so far unknown, gives the density and phase of the alternating current-sheet. But the total electric force $(E_0+E_1)e^{i\omega t}$, being parallel to the perfectly conducting screen, must vanish on the screen. Hence $f(\xi)$ must satisfy the integral equation

$$\int\limits_0^\infty f(\xi)H_0^{(2)}(k|x-\xi|)\,d\xi = g(x)$$

* *Math. Ann.* 47 (1896), 317. † *Electric Waves* (Cambridge, 1902).

‡ *Proc. London Math. Soc.* (2) 4 (1906), 190.

§ See, for example, Baker and Copson, *Huygens' Principle* (Oxford, 1939), chap. iv.

|| *Zeitschrift f. Phys.* 117 (1941), 168.

when $x > 0$, the function $g(x)$ being the known expression to which $-E_0$ reduces on the screen.

My interest in Magnus's work was aroused because I had already obtained independently the same integral equation by a different method and in a somewhat different connexion, though at the time I was unable to solve the integral equation. My work was concerned with the diffraction of plane sound waves by a perfectly reflecting half-plane, or its analytical equivalent, the diffraction of plane polarized electromagnetic waves in which the magnetic vector is parallel to the edge of the screen. In this problem the boundary condition is the vanishing of the normal derivative of the velocity potential (or magnetic force) on the screen. By the argument given in § 2 below I was led to Magnus's integral equation with the important difference—the screen is now the half-plane $y = 0$, $x < 0$.

Magnus asserts that, on physical grounds, one would expect the current function $f(\xi)$ to behave like a multiple of $\xi^{-\frac{1}{2}}$ when ξ is small, and he therefore assumes that

$$f(\xi) = \sum_{m=0}^{\infty} C_m \xi^{-1} J_{m+\frac{1}{2}}(\xi).$$

He then expands the function $g(x)$ as a series of the form

$$g(x) = \sum_{n=0}^{\infty} A_n J_n(x),$$

and obtains a set of linear equations for the infinite set of coefficients* C_m. Having found the coefficients C_m, Magnus ultimately arrives at the Sommerfeld solution for normal incidence after long and complicated analysis.

If this were the only method of solving the integral equation, the new solution of the Sommerfeld problem would have little merit—the introduction of the integral equation would in no way simplify the problem. If series of Bessel functions are to be used, it is simpler to use them from the beginning as Macdonald did. It turns out, however, that it is possible to give a direct solution of the integral equation by means of complex Fourier transforms and the theory of functions of a complex variable. This solution is given in § 3 of the present paper, and the deduction of Sommerfeld's formula follows in § 4.

* The set of linear equations is one which had been solved rigorously by Titchmarsh, *Math. Zeits.* 25 (1926), 321, though Magnus appears to be unacquainted with this work.

2. The derivation of the integral equation

Let us consider monochromatic sound waves of complex velocity-potential $\phi e^{i\omega t}$, where ϕ is independent of t and $\omega > 0$. The time-factor $e^{i\omega t}$ will be omitted hereafter. Then the velocity-potential ϕ satisfies the partial differential equation

$$\nabla^2\phi + k^2\phi = 0, \tag{2.1}$$

where $k = \omega/c$ and c is the velocity of sound. If plane waves with velocity-potential

$$\phi_0 = e^{ikx\cos\alpha + iky\sin\alpha} \quad (0 < \alpha < \pi)$$

are incident on the rigid screen $y = 0$, $x < 0$, the total velocity-potential will be of the form

$$\phi = \phi_0 + \phi_1,$$

where ϕ_1 represents the waves scattered by the obstacle. (See Fig. 1.)

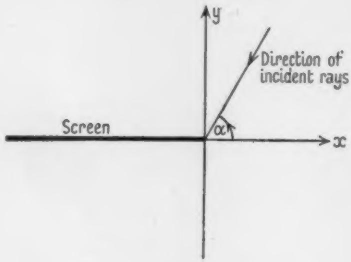

FIG. 1

The function ϕ_1 will be a function of x and y alone which satisfies the following conditions:

(i) ϕ_1 is a solution of (2.1) which is continuous and has continuous partial derivatives of the first and second orders in the (x, y)-plane supposed cut along the negative part of the x-axis;

(ii) the normal derivative of $\phi_0 + \phi_1$ vanishes on $y = 0$, $x < 0$;

(iii) at infinity, ϕ_1 satisfies Sommerfeld's radiation condition.*

These conditions determine ϕ_1 uniquely. The radiation condition is essential; for otherwise it would be possible to add to the solution terms representing standing waves.

* Sommerfeld, *Jahresbericht der D.M.V.* 21 (1912), 309. Cf. Baker and Copson, loc. cit., 25.

In its original form the radiation condition is difficult to apply. A simpler form is to suppose that k is a complex number with small negative imaginary part so that

$$-\tfrac{1}{2}\pi < \arg k < 0 \tag{2.2}$$

and to require that ϕ_1 vanish at infinity. Roughly speaking, this form of the radiation condition prevents ϕ_1 from containing any terms proportional to e^{ikr}, so that ϕ_1 represents only expanding waves at infinity. The condition (2.2) has a simple physical interpretation; for sound waves it means that the air has slight viscosity, whereas for electromagnetic waves, it asserts that the medium has slight conductivity. An additional advantage of this form of the condition is that, when k is complex, the Fourier-transform analysis is greatly simplified.

Now a solution of (2.1) which is continuous and has continuous derivatives of the first and second orders in the half-space $y > 0$ and which satisfies there the radiation condition is given by the formula*

$$\phi(x,y,z) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left[\frac{\partial}{\partial\eta}\phi(\xi,\eta,\zeta)\right]_{\eta=0}\frac{e^{-ikr}}{r}\,d\xi d\zeta, \tag{2.3}$$

where $$r^2 = (x-\xi)^2+y^2+(z-\zeta)^2,$$

in the half-space $y > 0$. In $y < 0$, the sign of the right-hand side is reversed. By condition (ii),

$$\frac{\partial\phi_1}{\partial y} = -ik\sin\alpha\, e^{ikx\cos\alpha}+f(x) \tag{2.4}$$

when $y = 0$, where $f(x)$ is identically zero when $x < 0$ but is unknown when $x > 0$. Hence we have

$$\phi_1 = \mp\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{-ik\sin\alpha\, e^{ik\xi\cos\alpha}+f(\xi)\}\frac{e^{-ikr}}{r}\,d\xi d\zeta$$

$$= \pm\tfrac{1}{2}i\int\limits_{-\infty}^{\infty}\{-ik\sin\alpha\, e^{ik\xi\cos\alpha}+f(\xi)\}H_0^{(2)}(kR)\,d\xi,$$

where $$R = +\sqrt{\{(x-\xi)^2+y^2\}};$$

* See, for example, Rayleigh, *Sound*, 2 (1896), 107, equation (3). The formula is easily proved by the theory of Green's function. In our problem, ϕ happens to be independent of z.

the upper or lower sign is taken according as y is positive or negative. It follows that

$$\phi = e^{ikx\cos\alpha+iky\sin\alpha} \pm \tfrac{1}{2}i \int_{-\infty}^{\infty} \{-ik\sin\alpha\, e^{ik\xi\cos\alpha} + f(\xi)\} H_0^{(2)}(kR)\, d\xi. \quad (2.5)$$

If $f(\xi)$ were identically zero, the problem would be the reflection of plane waves at a perfectly reflecting plane $y = 0$, and ϕ would be

$$e^{ikx\cos\alpha+iky\sin\alpha} + e^{ikx\cos\alpha-iky\sin\alpha}$$

for $y > 0$, and zero for $y < 0$. In our problem $f(\xi)$ is identically zero only for $x < 0$; hence (2.5) becomes

$$\phi = e^{ikx\cos\alpha+iky\sin\alpha} + e^{ikx\cos\alpha-iky\sin\alpha} + \tfrac{1}{2}i \int_0^{\infty} f(\xi) H_0^{(2)}(kR)\, d\xi \quad (2.6)$$

for $y > 0$, and
$$\phi = -\tfrac{1}{2}i \int_0^{\infty} f(\xi) H_0^{(2)}(kR)\, d\xi \quad (2.7)$$

for $y < 0$.

The velocity-potential ϕ is continuous across the aperture of the screen, that is, across the half-plane $y = 0$, $x > 0$. It follows from (2.6) and (2.7) that $f(\xi)$ satisfies the integral equation

$$\int_0^{\infty} f(\xi) H_0^{(2)}(k|x-\xi|)\, d\xi = 2ie^{ikx\cos\alpha} \quad (2.8)$$

when $x > 0$. We have thus proved

THEOREM 1. *When plane sound waves with velocity-potential* $e^{ikx\cos\alpha+iky\sin\alpha}$ $(0 < \alpha < \pi)$, *are incident on the perfectly reflecting half-plane* $y = 0$, $x < 0$, *the total velocity-potential of incident and scattered waves is given by equations* (2.6) *and* (2.7), *where* $f(\xi)$ *is the solution of the integral equation* (2.8).

In the problem considered by Magnus, plane polarized electromagnetic waves, with electric force

$$E_0 = e^{ikx\cos\alpha+iky\sin\alpha}$$

parallel to Oz, are incident on the perfectly reflecting half-plane $y = 0, x > 0$. The integral equation for this problem can be obtained by a similar argument. For the electric force E_1 in the scattered waves is parallel to Oz and satisfies the following conditions:

(i) E_1 is a solution of (2.1), depending on x and y alone, which is continuous and has continuous derivatives of the first and second

orders in the (x, y)-plane supposed cut along the positive part of the axis of x;

(ii) $E_1 = -E_0$ on $y = 0$, $x > 0$;

(iii) E_1 satisfies Sommerfeld's radiation condition at infinity.

It follows that E_1 is an even function of y, $\partial E_1/\partial y$ an odd function.

Hence
$$\frac{\partial E_1}{\partial y} \to \pm f(x) \quad \text{as} \quad y \to \pm 0.$$

But $\partial E_1/\partial y$ is continuous when $y = 0$, $x < 0$; therefore $f(x) = 0$ when $x < 0$. Hence we have, by (2.3),

$$E_1 = -\frac{1}{2\pi}\int\limits_0^\infty \int\limits_{-\infty}^\infty f(\xi)\frac{e^{-ikr}}{r}\,d\xi d\zeta = \tfrac{1}{2}i\int\limits_0^\infty f(\xi)H_0^{(2)}(kR)\,d\xi.$$

But since $E_1 = -E_0$ when $y = 0$, $x > 0$, we again obtain the integral equation (2.8).

3. Solution of the Integral Equation

Magnus's integral equation is of the form

$$\int\limits_0^\infty f(\xi)H_0^{(2)}(k|x-\xi|)\,d\xi = g(x), \tag{3.1}$$

where $g(x) = 2ie^{ikx\cos\alpha}$ when $x > 0$. Although this equation appears to be of the type

$$\int\limits_{-\infty}^\infty f(\xi)l(x-\xi)\,d\xi = g(x) \tag{3.2}$$

discussed by Titchmarsh,* it is actually quite different. In Titchmarsh's work $g(x)$ is given for all x, and $f(x)$ has to be found for all x; here $g(x)$ is given for $x > 0$ and $f(x)$ is identically zero for $x < 0$, and we have to find $g(x)$ for $x < 0$ and $f(x)$ for $x > 0$.

In order to apply the theory of generalized Fourier integrals, it is necessary to make certain assumptions regarding the unknown function $f(x)$. By (2.4), we have, when $x > 0$,

$$f(x) = ik\sin\alpha\, e^{ikx\cos\alpha} + \psi(x), \tag{3.3}$$

where
$$\psi(x) = \lim_{y\to 0}\frac{\partial\phi_1}{\partial y}.$$

Since $-\partial\phi_1/\partial y$ is the velocity parallel to Oy in the scattered waves, we might assume on physical grounds that $\psi(x)$ is continuous and

* Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), 314–15.

differentiable in $x > 0$, tends to zero as $x \to \infty$ by the radiation condition, and tends to infinity as $x \to 0$ since the velocity is infinite at the edge of the screen. Magnus assumed that $\psi(x)$ becomes infinite like $x^{-\frac{1}{2}}$ as $x \to 0$. Actually it suffices to assume that $e^{-cx}\psi(x)$ *belongs to* $L(0,\infty)$ *for any positive value of* c. It follows that, if we write

$$k = p - iq \quad (p, q > 0), \tag{3.4}$$

$e^{-cx}f(x)$ belongs to $L(0,\infty)$ for $c > q\cos\alpha$.

If we define $g(x)$ for negative values of x by (3.1), it follows from our assumption concerning $\psi(x)$ that $g(x)$ is continuous for $x \leqslant 0$ and differentiable for $x < 0$. Moreover, since*

$$H_0^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{1}{4}\pi)},$$

where $|z|$ is large and $-2\pi < \arg z < \pi$, we have

$$H_0^{(2)}(k|x-\xi|) = O\{|x-\xi|^{-\frac{1}{2}} e^{-q|x-\xi|}\},$$

and so, when $x = -x'$, and x' is large and positive,

$$g(x) = o\int_0^\infty \{e^{q\xi\cos\alpha} + |\psi(\xi)|\} e^{-q(x'+\xi)}\,d\xi.$$

Therefore $g(x)$ tends to zero as $x \to -\infty$, and also $e^{cx}g(x)$ belongs to $L(-\infty,0)$ and to $L^2(-\infty,0)$ for any positive† value of c. This is what we should expect on physical grounds; for, comparing (3.1) and (2.7), we see that

$$g(x) = 2i \lim_{y\to -0} \phi(x,y).$$

I next state certain results needed in the course of the solution of the integral equation. The complex Fourier transform of the kernel $l(x) = H_0^{(2)}(k|x|)$ is denoted by

$$L(w) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} H_0^{(2)}(k|x|) e^{iwx}\,dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty H_0^{(2)}(kx) \cos wx\,dx.$$

Lemma 1. *$L(w)$ is an analytic function of the complex variable $w = u+iv$, regular in the strip $-q < v < q$, and has the form*

$$L(w) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{\sqrt{(k^2-w^2)}},$$

where the branch of the square root reduces to k when $w = 0$. The

* Watson, *Theory of Bessel Functions* (Cambridge, 1922), 198.

† Or even for $c > -q$, though this is unnecessary in what follows.

analytical continuation of $L(w)$ is regular over the whole plane supposed cut radially outwards from $w = k$ to infinity and from $w = -k$ to infinity.

To prove the formula for $L(w)$, let us suppose in the first instance that $|w| < |k|$. Let us write $\phi = \arg w$, $\gamma = -\arg k$ so that $0 < \gamma < \frac{1}{2}\pi$. We may then rotate the path of integration through an angle $-(\frac{1}{2}\pi - \gamma)$, to obtain

$$\sqrt{(\tfrac{1}{2}\pi)}\,L(w) = \int\limits_0^{\infty\exp(-\frac{1}{2}\pi i+\gamma i)} \cos wx\, H_0^{(2)}(kx)\,dx;$$

for, on the circular arc $z = Re^{-\theta i}$, where $0 < \theta < \frac{1}{2}\pi - \gamma$, we have

$$|\cos wz| \leqslant \exp\{R|w|\,.\,|\sin(\theta-\phi)|\},$$
$$|H_0^{(2)}(kz)| = o[\exp\{-R|k|\,.\,|\sin(\theta+\gamma)|\}],$$

and so, by Jordan's lemma, the integral along this arc tends to zero as $R \to \infty$. If we now put $kx = -it$, we have

$$\sqrt{\left(\frac{\pi}{2}\right)}L(w) = \int\limits_0^\infty \cos\left(\frac{iwt}{k}\right) H_0^{(2)}(-it)\frac{dt}{ik}$$
$$= \frac{2}{\pi k}\int\limits_0^\infty \cos\left(\frac{iwt}{k}\right) K_0(t)\,dt$$
$$= \frac{1}{\sqrt{(k^2-w^2)}}$$

by a known integral,* valid since

$$\left|\mathrm{Im}\left(\frac{iw}{k}\right)\right| \leqslant \frac{|w|}{|k|} < 1.$$

The rest of Lemma 1 then follows by analytical continuation.

The generalized Fourier integrals† we shall use are functions of the complex variable w, defined by

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty f(x)e^{iwx}\,dx,$$

$$F_-(w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^0 f(x)e^{-iwx}\,dx;$$

and similarly for $G_+(w)$ and $G_-(w)$. In the present problem $F_-(w)$ is identically zero.

* Watson, loc. cit. 388 (10). † Titchmarsh, loc. cit. 4.

LEMMA 2. *$F_+(w)$ is an analytic function of w, regular when $v > q\cos\alpha$. It is of the form*

$$F_+(w) = -\frac{1}{\sqrt{(2\pi)}}\frac{k\sin\alpha}{w+k\cos\alpha}+\Psi(w),$$

where $\Psi(w)$ is regular in $v > 0$, bounded in $v \geqslant c > 0$.

The formula for $F_+(w)$ follows from (3.3), with

$$\Psi(w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \psi(x)e^{iwx}\,dx.$$

Since $e^{-cx}\psi(x)$ belongs, by hypothesis, to $L(0,\infty)$ for any positive value of c, this last integral converges uniformly with respect to w in $v \geqslant c$. Hence $\Psi(w)$ is regular in $v > 0$. Moreover, if $v \geqslant c$,

$$|\Psi(w)| \leqslant \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty |\psi(x)e^{-cx}|\,dx.$$

LEMMA 3.

$$G_+(w) = -\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{w+k\cos\alpha} \quad \textit{when } v > q\cos\alpha.$$

This is evident, since

$$g(x) = 2ie^{ikx\cos\alpha}$$

when $x > 0$.

LEMMA 4. *$G_-(w)$ is an analytic function of w, regular in $v < 0$, and tends to zero as $u \to \pm\infty$ for any fixed negative value of v.*

Since $e^{cx}g(x)$ belongs to $L(-\infty, 0)$ for any fixed positive value of c, the integral defining $G_-(w)$ converges uniformly with respect to w in $v \leqslant -c$. Hence $G_-(w)$ is regular in $v < 0$. Moreover,

$$G_-(u-ic) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^0 e^{iux}g(x)e^{cx}\,dx,$$

which tends to zero as $u \to \pm\infty$, by the Riemann-Lebesgue theorem.*

LEMMA 5. *If $a > q\cos\alpha$, $b > 0$,*

$$g(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty+ai}^{\infty+ai} G_+(w)e^{-ixw}\,dw+\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty-bi}^{\infty-bi} G_-(w)e^{-ixw}\,dw$$

in the mean-square sense.

* Titchmarsh, loc. cit. 11.

The function $h(x)$, which is equal to $e^{-ax}g(x)$ when $x > 0$, zero when $x < 0$, belongs to $L^2(0,\infty)$. Its Fourier transform is

$$H(u) = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty g(x)e^{-ax}e^{iux}\,dx = G_+(u+ia).$$

Hence
$$h(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} G_+(u+ia)e^{-ixu}\,du$$

in the mean-square sense. Similarly, if $k(x)$ is equal to $e^{bx}g(x)$ for $x < 0$, zero for $x > 0$,

$$k(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} G_-(u+ia)e^{-ixu}\,du,$$

again in the mean-square sense. The result follows by addition.

Lemma 6. *If $q\cos\alpha < a < q$,*

$$\int_0^\infty f(\xi)H_0^{(2)}(k|x-\xi|)\,d\xi = \int_{-\infty+ia}^{\infty+ia} F_+(w)L(w)e^{-ixw}\,dw$$

in the mean-square sense.

The function $h(x) = e^{-ax}H_0^{(2)}(k|x|)$ belongs to $L^2(-\infty,\infty)$ if $-q < a < q$. Its Fourier transform is

$$H(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} H_0^{(2)}(k|x|)e^{-ax}e^{iux}\,du = L(u+ia).$$

The function $k(x) = e^{-ax}f(x)$, where $f(x) \equiv 0$ when $x < 0$, belongs to $L(-\infty,\infty)$ if $a > q\cos\alpha$. Its Fourier transform is

$$K(u) = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty f(x)e^{-ax}e^{iux}\,dx = F_+(u+ia).$$

The result now follows from a known theorem.*

Lemma 7. *Let $H(w)$ be regular in the strip $a_1 \leqslant v \leqslant a_2$. Let $H(u+iv)$ be $L(-\infty,\infty)$ or $L^2(-\infty,\infty)$ and tend to zero as $u \to \pm\infty$ for v in the above strip. Let $K(w)$ have similar properties in $b_1 \leqslant v \leqslant b_2$, where $b_2 < a_1$. Let*

$$\int_{-\infty+ia}^{\infty+ia} H(w)e^{-ixw}\,dw = \int_{-\infty+ib}^{\infty+ib} K(w)e^{-ixw}\,dw \tag{3.5}$$

* Titchmarsh, loc. cit. 90, Theorem 65.

for all x, where $a_1 < a < a_2$, $b_1 < b < b_2$. Then $H(w)$ and $K(w)$ are regular and equal in the strip $b_1 < v < a_2$ and tend to zero as $u \to \pm\infty$ uniformly in any interior strip.

This is a known result.*

We now apply these lemmas to the solution of the integral equation

$$\int_0^\infty f(x) H_0^{(2)}(k|x-\xi|)\, d\xi = g(x).$$

By Lemmas 5 and 6 we have, if $q\cos\alpha < a < q$, $b > 0$,

$$\int_{-\infty+ia}^{\infty+ia} \{\sqrt{(2\pi)}F_+(w)L(w) - G_+(w)\} e^{-ixw}\, dw = \int_{-\infty-ib}^{\infty-ib} G_-(w) e^{-ixw}\, dw$$

in the mean-square sense.

In the strip $q\cos\alpha+\epsilon \leqslant v \leqslant q-\epsilon$ $(\epsilon > 0)$, the function

$$H(w) = \sqrt{(2\pi)}F_+(w)L(w) - G_+(w)$$

is regular, and $H(u+iv)$ tends to zero as $u \to \pm\infty$ for v in the above strip; this follows from Lemmas 1, 2, 3 and the fact that $\Psi(w)$ is bounded. Moreover, $H(u+iv)$ belongs to $L^2(-\infty,\infty)$ for any field v in the strip.

In the strip $-b' \leqslant v \leqslant -\epsilon$ $(\epsilon,\ b' > 0)$, $G_-(w)$ is regular and $G_-(u+iv)$ tends to zero as $u \to \pm\infty$ for v in the strip. Lastly, since $e^{cx}g(x)$ belongs to $L^2(-\infty, 0)$ for any positive c, $G_-(u+iv)$ belongs to $L^2(-\infty,\infty)$ for any fixed v in the strip.

The conditions of Lemma 7 are thus all satisfied. It follows that $H(w)$ and $G_-(w)$ can both be continued analytically throughout the strip $-b < v < q$ and are equal and regular there; moreover, $H(u+iv)$ and $G_-(u+iv)$ tend to zero uniformly as $u \to \pm\infty$ in any interior strip.

But $G_-(w)$ is regular in $v \leqslant -b$ for any positive value of b. Hence we can continue $H(w)$ all over the half-plane $v < q$, and so

$$\sqrt{(2\pi)}F_+(w)L(w) - G_+(w) = G_-(w)$$

when $v < q$.

By Lemmas 1 and 3 we now have

$$F_+(w) = \tfrac{1}{2}\sqrt{(k^2-w^2)}\left\{G_-(w) - \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{w+k\cos\alpha}\right\}. \tag{3.6}$$

Lemma 2 shows that $F_+(w)$ is regular in $v > 0$ apart from a simple pole at $w = -k\cos\alpha$. By (3.6), $F_+(w)$ can therefore be continued

* Titchmarsh, loc. cit. 255, Theorem 141.

analytically all over the w-plane, and the only singularities of the function so defined are the above simple pole and a branch-point at $w = k$ in the lower half-plane. We may therefore write

$$F_+(w) = \frac{\sqrt{(k-w)}}{w+k\cos\alpha}\,\phi(w), \tag{3.7}$$

where $\phi(w)$ is an integral function.

Substituting this value of $F_+(w)$ in (3.6), we obtain

$$\phi(w) = \tfrac{1}{2}\sqrt{(k+w)}\left\{-\sqrt{\left(\frac{2}{\pi}\right)}+(w+k\cos\alpha)G_-(w)\right\}. \tag{3.8}$$

Since $G_-(w)$ is regular at $w = -k\cos\alpha$, it follows that

$$\phi(-k\cos\alpha) = -\frac{\sqrt{(k-k\cos\alpha)}}{\sqrt{(2\pi)}},$$

and so
$$\phi(w) = -\frac{\sqrt{(k-k\cos\alpha)}}{\sqrt{(2\pi)}}+(w+k\cos\alpha)\chi(w),$$

where $\chi(w)$ is also an integral function. Hence we have, by (3.7) and (3.8),

$$F_+(w) = -\frac{\sqrt{(k-k\cos\alpha)}\sqrt{(k-w)}}{\sqrt{(2\pi)}(w+k\cos\alpha)}+\sqrt{(k-w)}\chi(w), \tag{3.9}$$

$$G_-(w) = \sqrt{\left(\frac{2}{\pi}\right)}\left\{1-\frac{\sqrt{(k-k\cos\alpha)}}{\sqrt{(k+w)}}\right\}\frac{1}{w+k\cos\alpha}+\frac{2\chi(w)}{\sqrt{(k+w)}}. \tag{3.10}$$

It remains to determine $\chi(w)$.

Now $F_+(w)$ is bounded in $v \geqslant c$, where $q\cos\alpha < c < q$; hence, by (3.9), $\chi(w)$ is $O(w^{-\frac{1}{2}})$ as $|w| \to \infty$ in $v \geqslant c$. Again $G_-(w)$ is bounded in $v \leqslant -c$, and so, by (3.10), $\chi(w)$ is $O(w^{\frac{1}{2}})$ as $|w| \to \infty$ in $v \leqslant -c$. Lastly $G_-(w)$ tends to zero as $u \to \pm\infty$ uniformly in $-c \leqslant v \leqslant c$; hence $\chi(w)$ is $o(w^{\frac{1}{2}})$ as $|w| \to \infty$ in this strip. Hence the integral function $\chi(w)$ is $O(w^{\frac{1}{2}})$ as $|w| \to \infty$. By the extension of Liouville's theorem* $\chi(w)$ is a polynomial of degree $\leqslant \frac{1}{2}$, and so is a constant; and, as $\chi(w)$ is $O(w^{-\frac{1}{2}})$ in $v \geqslant c$, this constant is zero. Hence we have

$$F_+(w) = -\frac{\sqrt{(k-k\cos\alpha)}\sqrt{(k-w)}}{\sqrt{(2\pi)}(w+k\cos\alpha)},$$

$$G_-(w) = \sqrt{\left(\frac{2}{\pi}\right)}\left\{1-\frac{\sqrt{(k-k\cos\alpha)}}{\sqrt{(k+w)}}\right\}\frac{1}{w+k\cos\alpha}.$$

We have thus proved

THEOREM 2. *Let $k = p-iq$, where $p > 0$, $q > 0$. Let $f(x)$ be of the form*

$$f(x) = ik\sin\alpha\, e^{ikx\cos\alpha}+\psi(x),$$

* Titchmarsh, *The Theory of Functions* (Oxford, 1932), 85.

where $e^{-cx}\psi(x)$ *belongs to* $L(0,\infty)$ *for any positive value of* c, *and* $0 < \alpha < \pi$. *Then the integral equation*

$$\int_0^\infty f(\xi)H_0^{(2)}(k|x-\xi|)\,d\xi = 2ie^{ikx\cos\alpha} \quad (x > 0)$$

has the solution whose complex Fourier transform is

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty f(x)e^{iwx}\,dx = -\frac{\sqrt{(k-k\cos\alpha)}\sqrt{(k-w)}}{\sqrt{(2\pi)}(w+k\cos\alpha)} \quad (3.11)$$

when $v > q\cos\alpha$. *The explicit form of the solution is*

$$f(x) = ik\sin\alpha\, e^{ikx\cos\alpha} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{(k-k\cos\alpha)}\sqrt{(k-w)}}{w+k\cos\alpha}e^{-ixw}\,dw. \quad (3.12)$$

Although it is not necessary to find $f(x)$ explicitly in the solution of the diffraction problem, it is nevertheless of some interest to give the actual solution of the integral equation (3.1). We have

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} F_+(w)e^{-ixw}\,dw \\ &= -\frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty}\frac{\sqrt{(k-k\cos\alpha)}\sqrt{(k-w)}}{w+k\cos\alpha}e^{-ixw}\,dw \end{aligned}$$

where $q\cos\alpha < a < q$. Deforming the path of integration into the real axis, we obtain the required formula.

If we make the imaginary part of $k = p-iq$ tend to zero, the equation (3.12) will still hold, provided that we indent the real axis, the indentation being downwards at $w = -k\cos\alpha$ and upwards at $w = k$.

4. The solution of Sommerfeld's problem

Instead of solving the diffraction problem by using equations (2.6) and (2.7) to get $\phi(x,y)$ from $f(x)$, it is much simpler to make use of the transform $F_+(w)$ in the following way.

When $y < 0$, we have, by (2.7),

$$\begin{aligned} \phi(x,y) &= -\tfrac{1}{2}i\int_0^\infty f(\xi)m(x-\xi)\,d\xi \\ &= -\tfrac{1}{2}i\int_{-\infty+ia}^{\infty+ia} F_+(w)M(w)e^{-ixw}\,dw, \end{aligned}$$

where $$m(x) = H_0^{(2)}\{k\sqrt{(x^2+y^2)}\}$$

$$M(w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} m(x)e^{ixw}\,dx$$

and $$q\cos\alpha < a < q.$$

The function $M(w)$ is regular in the strip $-q < v < q$; its value is*

$$M(w) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{\sqrt{(k^2-w^2)}}e^{-i|y|\sqrt{(k^2-w^2)}}$$

where the branch of the square root reduces to k when $w = 0$. The analytical continuation of $M(w)$ is regular in the whole w-plane supposed cut radially outwards from k to infinity and from $-k$ to infinity. Thus $\phi(x, y)$ is determined for $y < 0$; and similarly for $y > 0$. The resulting solution of Sommerfeld's diffraction problem is contained in the following theorem.

THEOREM 3. *When plane sound waves with velocity-potential* $e^{ikx\cos\alpha+iky\sin\alpha}$ $(0 < \alpha < \pi)$ *are incident on the perfectly reflecting half-plane* $y = 0$, $x < 0$, *the total velocity-potential of incident and scattered waves is given by*

$$\phi(x, y) = e^{ikx\cos\alpha+iky\sin\alpha}+e^{ikx\cos\alpha-iky\sin\alpha}+$$

$$+\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia} e^{-ixw-iy\sqrt{(k^2-w^2)}}\frac{\sqrt{(k-k\cos\alpha)}}{\sqrt{(k+w)}}\frac{dw}{w+k\cos\alpha}$$

when $y > 0$, *and by*

$$\phi(x, y) = -\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia} e^{-ixw+iy\sqrt{(k^2-w^2)}}\frac{\sqrt{(k-k\cos\alpha)}}{\sqrt{(k+w)}}\frac{dw}{w+k\cos\alpha},$$

when $y < 0$, *where* $q\cos\alpha < a < q$.

To identify this form of the solution with that given by Sommerfeld, let us consider the case when (x, y) lies inside the geometrical shadow, so that

$$x = -\rho\cos\theta, \qquad y = -\rho\sin\theta,$$

* This result is the particular case of Watson's equation—loc. cit. 416, equation (2)—with $\mu = -\frac{1}{2}$, $\nu = 0$, $a = ik$, $b = w$, $z = y$.

where $\rho = +\sqrt{(x^2+y^2)}$, $0 < \theta < \alpha$. By Cauchy's theorem the path of integration can be deformed into the hyperbolic path

$$w = -k\cos(\theta+i\tau) \quad (-\infty < \tau < \infty),$$

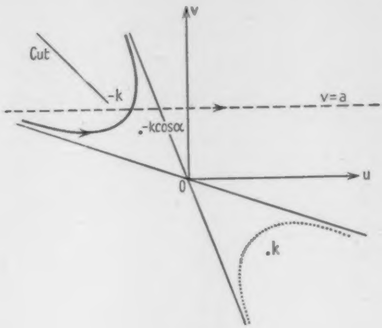

FIG. 2

without crossing the pole $w = -k\cos\alpha$ or the radial cut from $-k$ to infinity. (See Fig. 2.) This gives

$$\phi(x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ik\rho\cosh\tau}\frac{\sqrt{(1-\cos\alpha)}}{\sqrt{\{1-\cos(\theta+i\tau)\}}}\frac{\sin(\theta+i\tau)}{\cos(\theta+i\tau)-\cos\alpha}\,d\tau$$

$$= \frac{1}{4\pi}\int\limits_{-\infty}^{\infty} e^{-ik\rho\cosh\tau}\left\{\frac{1}{\sin\frac{1}{2}(\alpha-\theta-i\tau)}+\frac{1}{\sin\frac{1}{2}(\alpha+\theta+i\tau)}\right\}d\tau$$

$$= \frac{1}{4\pi}\int\limits_{0}^{\infty} e^{-ik\rho\cosh\tau}\left\{\frac{1}{\sin\frac{1}{2}(\alpha-\theta-i\tau)}+\frac{1}{\sin\frac{1}{2}(\alpha-\theta+i\tau)}+\right.$$

$$\left.+\frac{1}{\sin\frac{1}{2}(\alpha+\theta+i\tau)}+\frac{1}{\sin\frac{1}{2}(\alpha+\theta-i\tau)}\right\}d\tau$$

$$= \frac{1}{\pi}\int\limits_{0}^{\infty} e^{-ik\rho\cosh\tau}\left\{\frac{\sin\frac{1}{2}(\alpha-\theta)\cosh\frac{1}{2}\tau}{\cosh\tau-\cos(\alpha-\theta)}+\frac{\sin\frac{1}{2}(\alpha+\theta)\cosh\frac{1}{2}\tau}{\cosh\tau-\cos(\alpha+\theta)}\right\}d\tau$$

$$= \frac{1}{\pi}\int\limits_{0}^{\infty} e^{-ik\rho\cosh\tau}\left\{\frac{\cos\frac{1}{2}\gamma\cosh\frac{1}{2}\tau}{\cosh\tau+\cos\gamma}+\frac{\cos\frac{1}{2}\delta\cosh\frac{1}{2}\tau}{\cosh\tau+\cos\delta}\right\}d\tau$$

where $\qquad \gamma = \pi-\alpha+\theta, \qquad \delta = \pi-\alpha-\theta.$

This can be transformed into*

$$\phi(x,y)=\frac{e^{\pi i/4}}{\sqrt{\pi}}\left\{e^{-ik\rho\cos(\theta-\alpha)}\int_{-\infty}^{\sqrt{(2k\rho)}\sin\frac{1}{2}(\theta-\alpha)} e^{-i\lambda^2}\,d\lambda+\right.$$

$$\left.+e^{-ik\rho\cos(\theta+\alpha)}\int_{-\infty}^{-\sqrt{(2k\rho)}\sin\frac{1}{2}(\theta+\alpha)} e^{-i\lambda^2}\,d\lambda\right\}$$

which is Sommerfeld's result. The other cases can be handled in a similar manner.

* See, for example, Baker and Copson, loc. cit. 139-40.

ON TAC-INVARIANTS OF TWO CURVES IN A PROJECTIVE SPACE

By B. SEGRE (*Manchester*)

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LET $\mathscr{C}$, $\mathscr{C}'$ be two curves in a projective n-dimensional space S_n, and suppose that $\mathscr{C}$ and $\mathscr{C}'$ touch each other at a (possibly singular) point O, at which they have the same osculating spaces of dimensions $1, 2,..., r$ $(1 \leqslant r \leqslant n-1)$. When $r = n-1$, $\mathscr{C}$ and $\mathscr{C}'$ have (at O) $n-1$ projective tac-invariants $\mathscr{I}_1, \mathscr{I}_2,..., \mathscr{I}_{n-1}$, of which some simple geometric interpretations are known.* This has been extended to the case in which $1 < r < n-1$ by B. Su,† who has proved that $\mathscr{C}$ and $\mathscr{C}'$ have then (at O) $r-1$ projective tac-invariants

$$\mathscr{I}_1, \mathscr{I}_2,..., \mathscr{I}_{r-1}. \tag{1}$$

This extension can be deduced almost immediately from the previous result, by considering the projections $\mathscr{C}_1$, $\mathscr{C}_1'$ of $\mathscr{C}$, $\mathscr{C}'$ upon their common osculating S_r at O from an arbitrary S_{n-r-1} skew to S_r. For $\mathscr{C}_1$ and $\mathscr{C}_1'$ have at O the same osculating spaces of dimensions $1, 2,..., r-1$, and so the previous result can be applied. The $r-1$ projective tac-invariants, (1) say, thus arising from $\mathscr{C}_1$, $\mathscr{C}_1'$ at O, are easily seen to be independent of the centre S_{n-r-1} of projection, so that they also are projective tac-invariants of $\mathscr{C}$ and $\mathscr{C}'$ at O.

The last result gives nothing if $\mathscr{C}$ and $\mathscr{C}'$ touch at O but have distinct osculating planes at O, i.e. when $1 = r < n-1$. In this case, however, C. C. Hsiung has very recently obtained by direct calculation a number of projective tac-invariants, on the hypothesis that both $\mathscr{C}$ and $\mathscr{C}'$ have at O an ordinary simple point. Moreover, he has given an interesting, but rather intricate, geometric interpretation of these invariants, involving certain irrationalities. These results are now extended and completed by the following

THEOREM. *Let $\mathscr{C}$, $\mathscr{C}'$ be two curves in $S_n = S_n'$ $(n \geqslant 3)$ having an ordinary simple point O in common. Denote by S_k, S_k' their respective k-dimensional osculating spaces at O $(k = 1, 2,..., n-1)$, and suppose that*

$$S_1 = S_1', \quad S_2 = S_2', \quad ..., \quad S_r = S_r', \tag{2}$$

* Cf. B. Segre (**2**) also for further references on the subject.

† Cf. B. Su (**3**). I know this paper only through the review appearing in *Math. Reviews*, which contains no indication about the proof.

where r is any fixed integer satisfying $1 \leqslant r \leqslant n-2$, but that no other particularization of these spaces occurs. Hence, if $r+1 \leqslant i \leqslant n$, the intersection of S_{i-1} and S'_{n+r-i} is S_r, and not a space of higher dimension, so that the join of S_{i-1} and S'_{n+r-i} is a prime, which we indicate by ρ_i.

Let us fix in S_n an arbitrary prime π not containing O, and further consider a variable prime χ not containing O and tending to a position χ_0 through O, but not through S_1; hence χ meets the line S_1 at a point O_1 tending to O. The prime χ has then exactly one point in common with each of the curves $\mathscr{C}$, $\mathscr{C}'$ in the neighbourhood of O. We denote these two points by Q, Q', and the intersections of their join with the primes π, ρ_i by P, R_i respectively ($i = r+1, r+2,..., n$).

When χ tends to χ_0, the expression

$$\theta_{ij} = (QQ'R_i P)^{n+r-2j+1}/(QQ'R_j P)^{n+r-2i+1} \tag{3}$$

has a limit, which we denote by $\mathscr{I}_{ij}$ ($i \neq j$; $i, j = r+1, r+2,..., n$); this limit is independent of π, χ_0, as well as of the way in which χ tends to χ_0. Each of the numbers $\mathscr{I}_{ij}$ thus defined is consequently a projective tac-invariant of $\mathscr{C}$ and $\mathscr{C}'$ at O.

The tac-invariants $\mathscr{I}_{ij}$ satisfy the identities

$$\mathscr{I}_{ij} \times \mathscr{I}_{ji} = 1, \tag{4}$$

and, when $n \geqslant r+3$,

$$\mathscr{I}_{jh}^{n+r-2i+1} \times \mathscr{I}_{hi}^{n+r-2j+1} \times \mathscr{I}_{ij}^{n+r-2h+1} = 1, \tag{5}$$

where i, j, h are any three different numbers out of $r+1$, $r+2,..., n$. Hence, in any case, each of the tac-invariants $\mathscr{I}_{ij}$ can be expressed as a function of

$$\mathscr{I}_{r+1,r+2}, \mathscr{I}_{r+1,r+3},..., \mathscr{I}_{r+1,n}. \tag{6}$$

The $n-r-1$ tac-invariants (6) *are independent; moreover, when $r > 1$, they are also independent of the $r-1$ tac-invariants* (1) *previously considered.*

When $r = 1$, the tac-invariants $\mathscr{I}_{ij}$ coincide with those otherwise defined by C. C. Hsiung (**1**), who has dealt only with this particular case. It may be added that the geometric interpretation of the tac-invariants $\mathscr{I}_{ij}$ given above, is remarkably simpler than that obtained by this author for $r = 1$.

Proof. Let us introduce in S_n any system of non-homogeneous projective coordinates $x_1, x_2,..., x_n$ with origin at O, having the axes of $x_1, x_2,..., x_r$ as independent lines lying in the r spaces (2) respectively, and the axes of $x_{r+1}, x_{r+2},..., x_n$ as independent lines not lying in S_r, but lying in the $(r+1)$-dimensional spaces of intersection of

S_{r+1} and S'_n, S_{r+2} and S'_{n-1},..., S_n and S'_{r+1} respectively. Then $\mathscr{C}$ and $\mathscr{C}'$ can be represented parametrically by the equations

$$x_1 = t,\, x_2 = \{a_2 t^2\},\dots,\, x_r = \{a_r t^r\},\, x_{r+1} = \{a_{r+1} t^{r+1}\},\dots,\, x_n = \{a_n t^n\} \quad (7)$$

and

$$x_1 = t',\, x_2 = \{a'_2 t'^2\},\dots,\, x_r = \{a'_r t'^r\},\, x_{r+1} = \{a'_{r+1} t'^n\},\dots,\, x_n = \{a'_n t'^{r+1}\} \quad (8)$$

respectively, where $a_2, a_3,\dots, a_n, a'_2, a'_3,\dots, a'_n$ are $2n-2$ non-zero independent constants, and the brackets $\{\ \}$ denote that we have written only the terms of lower infinitesimal order in t or t'. When $r > 1$, the $r-1$ tac-invariants (1) of $\mathscr{C}$ and $\mathscr{C}'$ at O are

$$\mathscr{I}_1 = a_2/a'_2,\ \mathscr{I}_2 = a_3/a'_3,\ \dots,\ \mathscr{I}_{r-1} = a_r/a'_r. \quad (9)$$

It is immediately seen that the prime ρ_i, previously defined for $i = r+1, r+2,\dots, n$, has the equation

$$x_i = 0.$$

Moreover, we can take the equation of π in the form

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n = 1, \quad (10)$$

and the equation of χ in the form

$$x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n = \epsilon \quad (\epsilon \neq 0), \quad (11)$$

since neither π nor χ contains the origin O, and χ meets the x_1-axis at a point $O_1(\epsilon, 0,\dots, 0)$ lying in the neighbourhood of O, so that all the coordinates of this point must be finite. When χ varies, the coefficients $\epsilon, \gamma_2,\dots, \gamma_n$ in (11) also vary; and, when χ tends to χ_0, these coefficients have finite limits, the first of which is zero. The points Q and Q', i.e. the intersections of χ with $\mathscr{C}$ and $\mathscr{C}'$ respectively which lie in the neighbourhood of O, are given by (7) and (8), where the parameter t or t' has to be chosen in the neighbourhood of 0 in such a way as to satisfy (11); hence

$$\lim_{\chi\to\chi_0} \epsilon/t = 1, \qquad \lim_{\chi\to\chi_0} \epsilon/t' = 1. \quad (12)$$

We notice now that, when λ varies, the prime

$$x_i + \lambda(c_1 x_1 + c_2 x_2 + \dots + c_n x_n - 1) = 0 \quad (r+1 \leqslant i \leqslant n) \quad (13)$$

describes a pencil, containing ρ_i and π for $\lambda = 0$ and $\lambda = \infty$ respectively. The primes of this pencil passing through Q and Q' arise from certain values of λ, say $\lambda = \mu$ and $\lambda = \mu'$ respectively; from (7), (8), (13) we see that

$$\mu = \{a_i t^i\}, \qquad \mu' = \{a'_i t'^{n+r-i+1}\},$$

where the brackets { } have the same significance as before. It follows that

$$(QQ'R_i P) = (\mu\mu' 0\, \infty) = \mu/\mu' = \{a_i t^i\}/\{a_i' t'^{n+r-i+1}\},$$

since the cross-ratio of the points Q, Q', R_i, P equals the cross-ratio of the primes of the pencil (13) containing them, and so (3) gives

$$\theta_{ij} = [\{a_i t^i\}/\{a_i' t'^{n+r-i+1}\}]^{n+r-2j+1}/[\{a_j t^j\}/\{a_j' t'^{n+r-j+1}\}]^{n+r-2i+1}.$$

The right-hand side of this equation remains unaltered if we multiply the expressions in square brackets by $\epsilon^{n+r-2i+1} = \epsilon^{-i}/\epsilon^{-(n+r-i+1)}$ and $\epsilon^{n+r-2j+1} = \epsilon^{-j}/\epsilon^{-(n+r-j+1)}$ respectively. Then, from (12), we see that, when χ tends to χ_0, θ_{ij} has in fact the limit

$$\mathscr{I}_{ij} = (a_i/a_i')^{n+r-2j+1}/(a_j/a_j')^{n+r-2i+1}, \tag{14}$$

which does not depend on the coefficients of the equations (10), (11).

The projective invariance of $\mathscr{I}_{ij}$ is now an immediate consequence of the projective invariance of the cross-ratio. Moreover, (14) shows that the tac-invariants $\mathscr{I}_{ij}$ satisfy the identities (4), (5), and that the $n-r-1$ tac-invariants (6) are independent, because they have a non-zero Jacobian determinant with respect to the $n-r-1$ arguments $a_{r+2}, a_{r+3},..., a_n$. Finally, if $r > 1$, the tac-invariants (6) are independent of the tac-invariants (1), since the expressions (9) for the latter involve none of these $n-r-1$ arguments.

REFERENCES

1. C. C. Hsiung, 'Projective invariants of contact of two curves in space of n dimensions'. See below, pp. 39–45.
2. B. Segre, 'Sugli elementi curvilinei che hanno comuni le origini ed i relativi spazi osculatori': *Rend. dei Lincei*, (6) 22 (1935), 392–9.
3. B. Su, 'Note on a theorem of B. Segre': *Science Record, Ac. Sinica*, 1 (1942), 16–19.

PROJECTIVE INVARIANTS OF CONTACT OF TWO CURVES IN SPACE OF n DIMENSIONS

By C. C. HSIUNG (*Meitan*)

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1. Introduction

It is known that, if two plane curves touch at a point O, having contacts of order $k-1$ with their common tangent at O, and if non-homogeneous projective coordinates (x, y) are so chosen that the power-series expansions representing these two curves in the neighbourhood of O are $y = ax^k+...$ and $y = bx^k+...$, then the ratio a/b is a projective invariant of the curves, which was first found by H. J. S. Smith† and R. Mehmke‡ in the case $k = 2$. Next C. Segre gave§ the invariant in the general case, with a simple geometrical characterization, and extended|| the case $k = 2$ to two curves in ordinary space touching at a general point with common osculating plane. Then B. Segre¶ further extended this case to two curves in a projective space S_n of n dimensions having at a common point the same osculating linear spaces of dimensions $1,..., n-1$. Recently, as a supplement to B. Segre's investigation, B. Su** has studied two curves in space S_n having at a common point the same osculating linear spaces of dimensions $1,..., k$ $(1 < k \leqslant n-1)$, and characterized some projective invariants of the curves, using two theorems due to S. S. Chern.††

† H. J. S. Smith, 'On the focal properties of homographic figures': *Proc. London Math. Soc.* (1), 2 (1869), 196–248.

‡ R. Mehmke, 'Einige Sätze über die räumliche Collineation und Affinität welche sich auf die Krümmung von Kurven und Flächen beziehen': *Zeits. für Math. und Phys.* 36 (1891), 56–60; 'Über zwei die Krümmung von Kurven und das Gauss'sche Krümmungsmass von Flächen betreffende charakteristische Eigenschaften der linearen Punkttransformationen': ibid. 36 (1891), 206–13.

§ C. Segre, 'Su alcuni punti singolari delle curve algebriche, e sulla linea parabolica di una superficie': *Rend. dei Lincei* (5), 6 (1897), 168–75.

|| C. Segre, 'Sugli elementi curvilinei, che hanno comuni la tangente e il piano osculatore': ibid. (5), 33 (1924), 325–9.

¶ B. Segre, 'Sugli elementi curvilinei che hanno comuni le origini ed i relativi spazi osculatori': ibid. (6), 22 (1935), 392–9.

** B. Su, 'Note on a theorem of B. Segre': *Science Record, Ac. Sinica*, 1 (1942), 16–19.

†† S. S. Chern, 'Sur les invariants de contact dans la géométrie projective différentielle', to be published in Italy.

It is the purpose of the present paper to supplement the investigations of B. Segre and Su, by studying two curves in S_n having only one point and the tangent at it in common. This investigation may be regarded as a generalization of another one by the author.†

2. Derivation of invariants

Let C_1, C_2 be two curves in a projective space S_n touching at a general point O, and suppose that the osculating plane of either of them at O is not contained in the osculating hyperplane of the other at O. Further, let $x_1,\ldots, x_n$ represent non-homogeneous coordinates of a point in S_n. If we choose O as origin, the common tangent of C_1, C_2 at O as x_1-axis, the axes of x_2 and x_n in the osculating planes of C_1 and C_2 at O respectively, and the x_i-axis ($i = 3,\ldots, n-1$) in the plane of intersection of the osculating linear space of i dimensions of C_1 at O with the osculating linear space of $n-i+2$ dimensions of C_2 at O, then, in the neighbourhood of O, the equations of the two curves can be written in the form

$$C_1:\quad x_i = a_{ii}x_1^i+\ldots \quad (i = 2,\ldots, n), \tag{1}$$

$$C_2:\quad x_i = b_{i,n-i+2}x_1^{n-i+2}+\ldots \quad (i = 2,\ldots, n). \tag{2}$$

To find projective invariants of contact of C_1, C_2 at O, we consider the most general projective transformation of coordinates which leaves the point O and the common tangent invariant, and changes the other axes in such a way that the new axes have the same properties as the old. This transformation is expressed in terms of the non-homogeneous coordinates by the equations

$$\left.\begin{aligned} x_1 &= \Big(\sum_{k=1}^{n} \alpha_{1k}x_k^*\Big)\Big/\Big(1+\sum_{k=1}^{n}\alpha_{0k}x_k^*\Big) \\ x_i &= \alpha_{ii}x_i^*\Big/\Big(1+\sum_{k=1}^{n}\alpha_{0k}x_k^*\Big) \quad (i = 2,\ldots, n) \end{aligned}\right\}. \tag{3}$$

The effect of this transformation on (1), (2) is to produce two other systems of equations of the same form whose coefficients, indicated by asterisks, are given by the formulae

$$\left.\begin{aligned} \alpha_{ii}a_{ii}^* &= \alpha_{11}^i a_{ii} \\ \alpha_{ii}b_{i,n-i+2}^* &= \alpha_{11}^{n-i+2}b_{i,n-i+2} \quad (i = 2,\ldots, n) \end{aligned}\right\}. \tag{4}$$

† C. C. Hsiung, 'Some projective invariants of certain pairs of space curves', to be published in America.

On eliminating the α's from the equations (4), we see immediately that *the expressions*

$$I_{ij} = \left(\frac{a_{ii}}{b_{i,n-i+2}}\right)^{n-2j+2} \Big/ \left(\frac{a_{jj}}{b_{j,n-j+2}}\right)^{n-2i+2} \quad (i \neq j;\ i,j = 2,\dots,n) \tag{5}$$

are projective invariants associated with the point O of contact of C_1, C_2.

In particular, if $j = n-i+2$ and if the space S_n is of odd dimensions, that is, if $n = 2m+1$, m being a positive integer, then from (5) we easily see that the expressions

$$J_{i,n-i+2} = \frac{a_{ii}\,a_{n-i+2,n-i+2}}{b_{i,n-i+2}\,b_{n-i+2,i}} \quad (i = 2,\dots, m+1) \tag{6}$$

are invariants. If, on the contrary, the space S_n is of even dimensions, that is, if $n = 2(m+1)$, then besides the invariants (6) there is the additional invariant

$$J_{m+2,m+2} = \frac{a_{m+2,m+2}}{b_{m+2,m+2}}. \tag{7}$$

3. Geometrical characterizations of the invariants $J_{i,n-i+2}$

Let the 'point at infinity' on the x_i-axis ($i = 1,\dots, n$) be denoted by O_i. Then, from § 1, we first obtain the following geometrical characterization:

The invariant $J_{m+2,m+2}$, associated with the point O of contact of the curves C_1, C_2 immersed in a space $S_{2(m+1)}$, is Segre's invariant of contact at O of the plane curves obtained by projecting C_1, C_2 from the space $O_2 \dots O_{m+1}\, O_{m+3} \dots O_{2(m+1)}$ on to the plane $OO_1\, O_{m+2}$.

In the second place, we shall characterize geometrically a general invariant $J_{i,n-i+2}$ at O of the curves C_1, C_2 immersed in a general space S_n. For this purpose we may assume, without loss of generality, that $2 \leqslant i < n-i+2$, and consider the principal plane† at O of the two projections Γ_1, Γ_2 of C_1, C_2 from the centre

$$O_2 \dots O_{i-1}\, O_{i+1} \dots O_{n-i+1}\, O_{n-i+3} \dots O_n$$

on to the space $OO_1\, O_i\, O_{n-i+2}$. It is immediately seen that the equations of the space $OO_1\, O_i\, O_{n-i+2}$ are

$$x_2 = \dots = x_{i-1} = x_{i+1} = \dots = x_{n-i+1} = x_{n-i+3} = \dots = x_n = 0, \tag{8}$$

and that the equations of Γ_1, Γ_2 are (8) and

$$\Gamma_1:\quad x_i = a_{ii}\, x_1^i + \dots, \qquad x_{n-i+2} = a_{n-i+2,n-i+2}\, x_1^{n-i+2} + \dots, \tag{9}$$

$$\Gamma_2:\quad x_i = b_{i,n-i+2}\, x_1^{n-i+2} + \dots, \qquad x_{n-i+2} = b_{n-i+2,i}\, x_1^i + \dots. \tag{10}$$

† G. H. Halphen, 'Sur les invariants différentiels des courbes gauches': *J. de l'École Pol.* 28 (1880), 25.

To obtain the principal plane we project Γ_1, Γ_2 on to the plane OO_1O_i from a point V arbitrarily chosen in the space $OO_1O_iO_{n-i+2}$. If

$$x_1=\bar{x}_1,\qquad x_i=\bar{x}_i,\qquad x_{n-i+2}=\bar{x}_{n-i+2},$$
$$x_2=\ldots=x_{i-1}=x_{i+1}=\ldots=x_{n-i+1}=x_{n-i+3}=\ldots=x_n=0$$

are the coordinates of V, the two projections are easily found to be given by equations (8), $x_{n-i+2}=0$ and

$$X_i=a_{ii}X_1^i+\ldots, \tag{11}$$

$$X_i=-b_{n-i+2,i}\frac{\bar{x}_i}{\bar{x}_{n-i+2}}X_1^i+\ldots \tag{12}$$

respectively. The cones projecting Γ_1, Γ_2 from V have contact of order at least i along OV if, and only if, the invariant of contact of the curves (11), (12) at O equals 1; then, and only then, the centre V of projection lies on the principal plane. Hence this plane has the equations (8) and

$$b_{n-i+2,i}x_i+a_{ii}x_{n-i+2}=0. \tag{13}$$

We now consider the system of cones of order $n-i+1$ and vertex O in the space $OO_1O_iO_{n-i+2}$ such that the polar surfaces of orders $1,\ldots,\ n+2i+1,\ n-2i+3,\ldots,\ n-i+1$ of any point in the plane OO_iO_{n-i+2} with respect to any cone (the last of these polars being the cone itself), all pass through the line OO_1. The equations of this system of cones can easily be written as (8) and

$$x_1^{n-2i+2}\sum_{\substack{j,k=0,1,\ldots,i-1\\ j+k=i-1}}\beta_{jk}x_i^jx_{n-i+2}^k+\sum_{\substack{j,k=0,1,\ldots,n-i+1\\ j+k=n-i+1}}\gamma_{jk}x_i^jx_{n-i+2}^k=0. \tag{14}$$

If the cones of this system have a contact of the $i(n-i+1)$th order with Γ_1 at O, then their equation (14) reduces to

$$a_{ii}^{n-i+1}x_1^{n-2i+2}x_{n-i+2}^{i-1}-a_{n-i+2,n-i+2}^{i-1}x_i^{n-i+1}+\sum_{\substack{j=0,1,\ldots,n-i\\ k=1,\ldots,n-i+1\\ j+k=n-i+1}}A_{jk}x_i^jx_{n-i+2}^k=0, \tag{15}$$

where the A's are arbitrary constants. Moreover, we determine other cones of this system by the conditions that they have contact of order $i(n-i+1)$ with Γ_2 at O. The equations of these cones are likewise found to be (8) and

$$b_{n-i+2,i}^{n-i+1}x_1^{n-2i+2}x_i^{i-1}-b_{i,n-i+2}^{i-1}x_{n-i+2}^{n-i+1}+\sum_{\substack{j=1,\ldots,n-i+1\\ k=0,1,\ldots,n-i\\ j+k=n-i+1}}B_{jk}x_i^jx_{n-i+2}^k=0, \tag{16}$$

where the B's are arbitrary constants. The cones (15), (16) intersect in n lines residually to OO_1. The planes projecting these n lines from

OO_1 are represented by the equation obtained on eliminating x_1 from (15) and (16); they may reduce to a single plane to be counted n times, which then coincides with one or other of the n planes represented by the equations (8) and

$$a_{n-i+2,n-i+2}^{i-1} b_{n-i+2,i}^{n-i+1} x_i^n - a_{ii}^{n-i+1} b_{i,n-i+2}^{i-1} x_{n-i+2}^n = 0. \qquad (17)$$

Thus we obtain the following geometrical characterization of the invariant $J_{i,n-i+2}$:

The cross-ratio of the planes OO_1O_{n-i+2}, OO_1O_i, (13), *and any one of the n planes* (17), *is equal to one of the values of* $-\sqrt[n]{J_{i,n-i+2}^{i-1}}$.

This characterization is simply an extension of the one already given for the invariant of two curves in ordinary space.†

4. A geometrical characterization of a general invariant I_{ij}

Finally, I shall give a geometrical characterization of a general invariant I_{ij} of the curves C_1, C_2 of S_n. Without loss of generality, it may be assumed that $2 \leqslant i < j$. On projecting the curves C_1, C_2 from the space $O_2 \ldots O_{i-1} O_{i+1} \ldots O_{j-1} O_{j+1} \ldots O_n$ on to the space $OO_1O_iO_j$, we obtain two curves K_1, K_2, given by the equations

$$x_2 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_{j-1} = x_{j+1} = \ldots = x_n = 0 \quad (18)$$

and

$$K_1\colon \quad x_i = a_{ii} x_1^i + \ldots, \qquad x_j = a_{jj} x_1^j + \ldots, \qquad (19)$$

$$K_2\colon \quad x_i = b_{i,n-i+2} x_1^{n-i+2} + \ldots, \qquad x_j = b_{j,n-j+2} x_1^{n-j+2} + \ldots. \qquad (20)$$

Firstly, as in the previous section, we consider in the space $OO_1O_iO_j$ a system of cones of order $j-1$ and vertex O, having contact of order $i(j-1)$ with the curve K_1 at O and satisfying the conditions that the polar surfaces of orders $1, \ldots, j-i-1, j-i+1, \ldots, j-1$ of any point in the plane OO_iO_j with respect to any of them, all pass through the line OO_1. The equations of this system are easily found to be (18) and

$$a_{ii}^{j-1} x_1^{j-i} x_j^{i-1} - a_{jj}^{i-1} x_i^{j-1} + \sum_{\substack{l=0,1,\ldots,j-2 \\ m=1,\ldots,j-1 \\ l+m=j-1}} E_{lm} x_i^l x_j^m = 0, \qquad (21)$$

where the E's are arbitrary constants. Similarly, we have in the space $OO_1O_iO_j$ another system of cones of order $n-i+1$ and vertex O, having contact of order $(n-i+1)(n-j+2)$ with the curve K_2 at O

† C. C. Hsiung, loc. cit.

and satisfying the conditions that the polar surfaces of orders

$$1,\ldots,\; j-i-1,\; j-i+1,\ldots,\; n-i+1$$

of any point in the plane $OO_i O_j$ with respect to any of them, all pass through the common tangent OO_1. The equations of this system are (18) and

$$b^{n-i+1}_{j,n-j+2}\, x_1^{j-i} x_i^{n-j+1} - b^{n-j+1}_{i,n-i+2}\, x_j^{n-i+1} + \sum_{\substack{l=1,\ldots,n-i+1\\ m=0,1,\ldots,n-i\\ l+m=n-i+1}} F_{lm}\, x_i^l x_j^m = 0, \tag{22}$$

where the F's are arbitrary constants. If we further impose on the cones (21), (22) the conditions that the planes through OO_1 in which their intersection lies reduce to a single plane to be counted n times, then these planes coincide with one or other of the n planes satisfying the equations (18) and

$$a_{jj}^{i-1}\, b^{n-i+1}_{j,n-j+2}\, x_i^n - a_{ii}^{j-1}\, b^{n-j+1}_{i,n-i+2}\, x_j^n = 0. \tag{23}$$

Next we consider in the space $OO_1 O_i O_j$ any algebraic surface of order $n(j-i)+n-i+1$, satisfying the following conditions:

(i) the polar surfaces of orders

$$1,\ldots,\; (n+1)(j-i)-1,\; (n+1)(j-i)+1,\ldots,\; (n+1)(j-i)+n-j+1$$

of any point in the plane $OO_i O_j$ with respect to the algebraic surface all pass through the point O_1;

(ii) the polar surfaces of orders

$$1,\ldots,\; n(j-i)-1,\; n(j-i)+1,\ldots,\; n(j-i)+n-i+1$$

of any point in the plane $O_1 O_i O_j$ all pass through the point O;

(iii) the curve of intersection of the surface with the plane $O_1 O_i O_j$ consists of $n(j-i)+n-i+1$ lines of which $(n+1)(j-i)$ coincide with $O_i O_j$, and the other $n-j+1$ coincide with $O_1 O_j$.

It is easily seen that the equations of any of these algebraic surfaces may be written as (18) and

$$x_1^{(n+1)(j-i)}\, x_i^{n-j+1} + \sum_{\substack{l,m=0,1,\ldots,n-i+1\\ l+m=n-i+1}} \delta_{lm}\, x_i^l x_j^m = 0. \tag{24}$$

We uniquely determine a surface in the system (24) by the condition of having a contact of the $j(n-i+1)$th order with K_1 at O. From (19) it follows at once that the equation (24) reduces now to

$$a_{jj}^{n-i+1}\, x_1^{(n+1)(j-i)}\, x_i^{n-j+1} - a_{ii}^{n-j+1}\, x_j^{n-i+1} = 0. \tag{25}$$

Similarly, we uniquely determine another algebraic surface of order $n(j-i)+j-1$ in the space $OO_1 O_i O_j$, satisfying the following conditions:

(i) the polar surfaces of orders

$$1,\ldots,\ (n+1)(j-i)-1,\ (n+1)(j-i)+1,\ldots,\ (n+1)(j-i)+i-1$$

of any point in the plane $OO_i O_j$ with respect to the algebraic surface all pass through the point O_1;

(ii) the polar surfaces of orders

$$1,\ldots,\ n(j-i)-1,\ n(j-1)+1,\ldots,\ n(j-i)+j-1$$

of any point in the plane $O_1 O_i O_j$ all pass through the point O;

(iii) the curve of intersection of the surface with the plane $O_1 O_i O_j$ consists of $n(j-i)+j-1$ lines of which $(n+1)(j-i)$ coincide with $O_i O_j$, and the other $i-1$ coincide with $O_1 O_i$;

(iv) the surface has contact of order $(j-1)(n-i+2)$ with K_2 at O.

The equations of this surface are found to be (18) and

$$b^{j-1}_{i,n-i+2} x_1^{(n+1)(j-i)} x_j^{i-1} - b^{i-1}_{j,n-j+2} x_i^{j-1} = 0. \tag{26}$$

The planes through OO_1 in which the curve of intersection of the two algebraic surfaces (25), (26) lies, are given by the equations (18) and

$$a_{jj}^{n-i+1} b^{i-1}_{j,n-j+2} x_i^n - a_{ii}^{n-j+1} b^{j-1}_{i,n-i+2} x_j^n = 0. \tag{27}$$

Thus we arrive at the following geometrical characterization of a general invariant I_{ij}:

The cross-ratio of the planes $OO_1 O_j$, $OO_1 O_i$, any one of the n planes (27), *and any one of the n planes* (23), *is equal to one of the values of* $\sqrt[n]{I_{ij}}$.

NOTE ON THE DISTRIBUTION OF THE INTERVALS BETWEEN PRIME NUMBERS

By LORD CHERWELL (*Oxford*)

[Received 29 April 1945]

In this note an attempt is made to set out some of the results concerning the frequency of prime-pairs, -triplets, and so forth which can be derived by probability methods from the assumption that the distribution of the prime numbers may be treated as 'random'. This approach clearly precludes any attempt at rigorous proof in the mathematical sense. Nevertheless, it seems interesting that results obtained by such simple means fit fairly closely the facts found by our (necessarily limited) enumeration from tables, and that in two cases at least formulae emerge which are identical, as has been pointed out to me, with those derived by more elaborate methods.*

If P is an odd prime number $P+n$ can only be prime if n is even. In what follows P and p always denote odd primes ($p \neq P$) and n always an even number.

If we assume a random distribution of primes, it is easy to show that the probability of $P+n$ being a prime is enhanced by a factor $(p-1)/(p-2)$ for each different odd prime p which divides n. Let us compare for instance the probability of $P+2p$ not being divisible by p with the probability of $P+2$ not being divisible by p. If we know nothing about P, the probabilities are clearly equal, namely, each $(p-1)/p$. But, if we know that P is a prime, then neither P nor $P+2p$ can be divisible by p.† On the other hand, since one of the $p-1$ intermediate odd numbers must be divisible by p, the chances of its not being $P+2$ are $(p-2)/(p-1)$. As the chance of $P+2p$ not

* I have to thank Professor E. Maitland Wright not only for criticizing this note and suggesting many improvements but also for drawing my attention to previous work in this field, viz. Hardy and Littlewood, *Acta Math.* 44 (1923), 1–70, and Staeckel, *Sitz. der Heidelberger Akad. der Wiss.*, Abt. 1916, No. 10, 1–45; 1917, No. 15, 1–52; 1918, No. 2, 1–47, and No. 14, 1–67, which also contain references to earlier work. A note on the connexion of this work with mine is at the end of this paper.

† If p were equal to P, this would of course not be true. As in our considerations P may always be taken as greater than $\frac{1}{2}n$, it cannot be a divisor of n. In any event the correction would be insignificant unless P were very small.

being divisible by p is 1, the probability, so far as the factor p is concerned, of $P+2p$ being prime, if P is prime, is

$$\frac{p-1}{p-2} = 1+\frac{1}{p-2}$$

times greater than the probability of $P+2$ being prime. Since this reasoning holds for any value or multiple of p, the formula is quite general, and we may write for the probability of $P+n$ being a prime if P is one, $W_n = W_2 \prod_{p|n,\, p\geq 3} (p-1/(p-2)$, the product being taken over all different odd prime divisors of n and W_2 being the probability that $P+2$ is a prime. It is of course clear from this formula that W_n has a minimum value W_2 when n is any power of 2.

In order to test this formula we may count how often the various intervals from 2 to, say, n_0 occur in a given region in the neighbourhood of N. According to the prime number theorem there will be about $\Delta N/\log N$ primes in the region ΔN. If n_0 is small compared with ΔN but large compared with $\log N$, there will be between each of these primes P and $P+n_0$ about $n_0/\log N$ primes. In the region ΔN therefore we shall count a total of something like $n_0\,\Delta N/(\log N)^2$ intervals. If the probability of finding an interval n is

$$W_n = W_2 \prod_{p|n,\, p>3} \frac{p-1}{p-2},$$

the total number of intervals counted must be equal to

$$\frac{\Delta N}{\log N} \sum_{n=2}^{n=n_0} W_2 \prod_{p|n,\, p>3} \left(1+\frac{1}{p-2}\right).$$

As there are $\frac{1}{2}n_0$ even numbers between 0 and n_0, this sum will have $\frac{1}{2}n_0$ terms, so that, if we denote the mean value of

$$\prod_{p|n,\, p>3} \left(1+\frac{1}{p-2}\right)$$

for all even intervals from 0 to n_0 by s_{n_0}, we have the equation

$$\frac{n_0 s_{n_0}}{2} \times W_2 \times \frac{\Delta N}{\log N} = \frac{n_0\,\Delta N}{(\log N)^2},$$

i.e.

$$W_2 = \frac{2}{s_{n_0} \log N},$$

from which it follows that $W_n = \dfrac{2}{s_{n_0} \log N} \displaystyle\prod_{p|n,\, p\geq 3} \left(1+\frac{1}{p-2}\right)$.

The mean value s_{n_0} of

$$\prod_{p|n}\left(1+\frac{1}{p-2}\right) = 1+\sum_{p|n}\frac{1}{p-2}+\sum_{p|n}\frac{1}{(p_1-2)(p_2-2)}\cdots$$

between 0 and n_0 may of course easily be found by summing all the $\frac{1}{2}n_0$ terms and dividing by $\frac{1}{2}n_0$. We have

$\left[\frac{n_0}{2}\right]$ terms 1 contributing to the average $\frac{n_0}{2}\times 1\times\frac{2}{n_0} = 1$;

$\left[\frac{n_0}{2p}\right]$ terms $\frac{1}{p-2}$ contributing to the average

$$\frac{n_0}{2p}\times\frac{1}{p-2}\times\frac{2}{n_0} = \frac{1}{p(p-2)};$$

$\left[\frac{n_0}{2p_1p_2}\right]$ terms $\frac{1}{(p_1-2)(p_2-2)}$ contributing to the average a further

$$\frac{n_0}{2p_1p_2}\times\frac{1}{(p_1-2)(p_2-2)}\times\frac{2}{n_0} = \frac{1}{p_1p_2(p_1-2)(p_2-2)};$$

and so on.

Hence, ignoring the square brackets, it is plain that s_{n_0} converges to a limit for large values of n_0.

$$S_2 = 1+\sum_{p>3}^{\infty}\frac{1}{p(p-2)}+\sum_{\substack{p>3\\ p_2>p_1}}^{\infty}\frac{1}{p_1p_2(p_1-2)(p_2-2)}\cdots$$

$$= \prod_{p>3}^{\infty}\left(1+\frac{1}{p(p-2)}\right) = 1\cdot 5147\ldots^{*}$$

each prime factor, of course, being counted only once so that p runs from 3 to ∞, p_2 is always greater than p_1, etc.

* It may be shown that the error is less than $(\log\frac{1}{2}n_0)^A/n_0$ where A is some suitable constant. For

$$\frac{1}{2}n_0\sum_{p\leqslant\frac{1}{2}n_0}\frac{1}{p-2}+\frac{2}{n_0}\sum_{p_1,p_2\leqslant\frac{1}{2}n}\frac{1}{(p_1-2)(p_2-2)}+\ldots$$

$$= \frac{2}{n_0}\prod_{p\leqslant\frac{1}{2}n_0}\left(1+\frac{1}{p-2}\right) \leqslant \frac{2}{n_0}\exp\left(\sum_{p\leqslant\frac{1}{2}n_0}\frac{1}{p-2}\right)$$

$$\leqslant \frac{2}{n_0}\exp(A\operatorname{loglog}\tfrac{1}{2}n_0) = \frac{2(\log\frac{1}{2}n_0)^A}{n_0}.$$

The error clearly tends to zero as $n_0\to\infty$. For the introduction of loglog $\frac{1}{2}n_0$, see Hardy and Wright, *Introduction to the Theory of Numbers*, Chapter 22, Theorem 422 or 429.

In the limit therefore the number of prime pairs of the form P, $P+2$,—and for that matter P, $P+4$, or P, $P+2^\nu$,—in an interval ΔN will be

$$F_2 = \frac{2}{S_2 \log N}\Delta\pi$$

if $\Delta\pi$ is the number of primes in the interval. If we put

$$\Delta\pi = \frac{\Delta N}{\log N}$$

this becomes
$$F_2 = \frac{2\Delta N}{S_2(\log N)^2} = \frac{1{\cdot}320...\Delta N}{(\log N)^2}.$$

The accuracy with which this formula—which is identical with that derived by Hardy and Littlewood and by Staeckel—represents the facts is altogether surprising. Indeed, as is shown in Table 1, it is more exact than the mere probability approach gives us any right to expect.

TABLE 1

$\Delta N . 10^{-3}$	$\Delta\pi$(obs.)	W_2 (calc.)	F_2 (calc.)	(F_2 obs.)	F_4 (obs.)	$F_2+F_4/2$ (obs.)
220 to 255	2,807	0·1065	299·5	295	299	297
90 to 110	1,735	0·1145	199	198	206	202
10 to 30	2,004	0·1333	267	262	267	264·5
5 to 10	562	0·1480	83·2	79	82	80·5
2 to 5	364	0·1619	59	65	59	62
0·5 to 2	207	0·1852	38·3	37	38	37·5
0 to 0·5	92	0·2392	22	23	24	23·5
Total			968	959	975	967

Throughout $\log N$ has been taken as the logarithm of $\frac{1}{2}(N_1+N_2)$, which is of course arbitrary. But we get almost equally good agreement if we use the expression

$$F_2 = \frac{2(\Delta\pi)^2}{S_2 \Delta N},$$

$\Delta\pi$ being the observed number of primes in the interval ΔN, or the expression

$$\frac{2\Delta N}{S_2(\log N)^2},$$

putting $\Delta\pi = \Delta N/\log N$.

Between 1,000,000 and 8,000,000 the agreement is almost equally striking as is shown in Table 1 (a). Here the calculated values are obtained from the formula $\frac{2\Delta N}{S_2(\log N)^2}$.

TABLE 1 (*a*)

	F_2 (calc.)	F_2 (obs.)*
1,000,000 – 1,100,000	686·6	725
2,000,000 – 2,100,000	625·0	644
6,000,000 – 6,100,000	541·4	545
7,000,000 – 7,100,000	531·5	525
8,000,000 – 8,100,000	522·3	518

A comparison given by Hardy and Littlewood of primes between 100,000 and 1,000,000 is equally satisfactory.

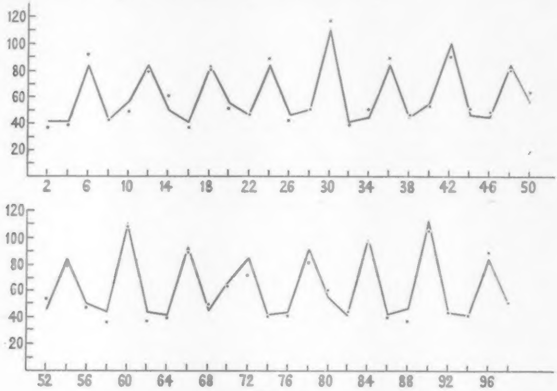

FIG. 1

The ordinates represent the frequencies of the various intervals which are plotted along the abscissa. The calculated frequencies are joined by straight lines; the observed frequencies are shown as separate dots.

To check the formula for W_n the frequencies of the intervals less than 100 between the primes in the region 250,000 to 255,000 are shown in Fig. 1 and Table 2. In actual calculations with a particular n_0 it is clearly best to use the value for s_{n_0} corresponding to that interval rather than S_2, which is the limiting value as $n_0 \to \infty$. The frequencies F_n for the various values of n are calculated by multiplying the total number of intervals counted by

$$\prod_{p|n,\, p>3} \frac{p-1}{p-2}$$

and dividing by

$$\sum_{n=2}^{n=98} \prod_{p|n,\, p>3} \frac{p-1}{p-2}.$$

The value of s_{n_0} for $n_0 = 98$ is 1·475.

* Glaisher, *Messenger of Math.* 8 (1879) 28–33.

TABLE 2

n	F_n (calc.)	F_n (obs.)	F_n (obs.) $-F_n$ (calc.)	$\sqrt{F_n}$
2	41·5	36	−5·5	6
4	41·5	38	−3·5	6·2
6	83	92	9	9·6
8	41·5	41	0·5	6·4
10	55·4	49	−6·4	7
12	83	79	−4	8·9
14	49·8	60	10·8	7·7
16	41·5	37	−4·5	6·1
18	83	82	−1	9·0
20	55·4	51	−4·4	7·1
22	46	46	0	6·8
24	83	88	4	9·4
26	46	42	−4	6·5
28	49·9	51	1·1	7·1
30	110·7	116	5·3	10·8
32	41·5	38	−3·5	6·2
34	44·3	51	6·7	7·1
36	83	88	5	9·4
38	43·9	46	2·1	6·8
40	55·4	53	−2·4	7·3
42	99·6	90	−9·6	9·5
44	46	51	5	7·1
46	43·5	48	4·5	6·9
48	83	80	3	9·0
50	55·4	63	7·6	8·0
52	45·3	53	7·7	7·3
54	83	80	−3	9·0
56	49·8	47	−2·8	6·8
58	43·1	36	−7·1	6
60	110·7	108	−2·7	10·4
62	42·9	37	−5·9	6·1
64	41·5	39	−2·5	6·2
66	92·1	88	−4·1	9·4
68	44·2	49	−5·8	7
70	66·5	64	−2·5	8
72	83	72	−11	8·5
74	42·7	40	−2·7	6·3
76	43·1	41	−2·1	6·4
78	91·8	81	−10·8	9
80	55·4	60	4·6	7·7
82	42·5	43	0·5	6·6
84	99·6	97	−2·6	9·9
86	42·4	39	−3·4	6·2
88	46	36	−10	6·0
90	110·7	105	−5·7	10·2
92	43·5	43	−0·5	6·5
94	42·4	41	−1·4	6·4
96	83	91	8	9·5
98	49·8	57	7·2	7·5

It will be observed that the mean deviation between observed and calculated frequencies is considerably less than $\sqrt{F_n}$.

A better check is perhaps obtained by comparing observed and calculated frequencies over groups of n containing the same prime divisors. Table 2 (a) shows the mean values calculated and observed for the more numerous groups of this kind starting with the even values of n containing only the divisor 3, then 3 and 5, etc.

TABLE 2 (a)

Odd prime divisors of n	3	3·5	3·7	5	7	11	0
Number	752	329	187	273	215	133	229
F_n (calc.)	83	110·7	99·6	55·4	53·7	46	41·5
F_n (obs.)	83·5	109·3	93·5	54·6	49·8	44·3	38·2
Δ (in. %)	0·6	−1·3	−6·1	−1·4	−6·5	−3·7	−8·0
$F_n^{-\frac{1}{2}}$ (in. %)	3·6	5·5	7·3	6·5	6·8	8·7	6·6

It is clear that, as the numbers go up, the percentage deviation goes down as it should do. The fact that the last column, which gives the frequency of the n's which are powers of 2 and contain no prime factors, is rather lower than calculated does not seem significant. Thus the intervals 2 and 4 which seem abnormally low in this count between 250,000 and 255,000, i.e. 36 and 38 instead of the calculated 41·5, are 48 and 46 in the region 245,000 to 250,000 and average 42·1 and 42·7 respectively for intervals of 5,000 over the range from 220,000 to 255,000.

Similar reasoning can be extended to more complicated cases. How for instance does the probability of $P+3p$ being prime compare with the probability of, say, $P+8$ being prime if we know that both P and $P+2$ are primes? If P and $P+2$ are primes, obviously $P+4$ and any number of the form $P+6\nu+4$ must be divisible by 3. Just as we have hitherto omitted odd values of n, which would automatically make $P+n$ even, we may in this section confine ourselves to a sequence of values of n of the form 6ν and $6\nu+2$. Clearly there will be $\frac{1}{3}$ as many terms in this sequence as there are whole numbers. But, as we need not bother about prime divisors of n which are less than 5, since numbers divisible by 2 and 3 have now been accounted for, it is clear that on the average one term in p in our sequence 6ν or $6\nu+2$, must be divisible by any prime p greater than 3. Between P and $P+3p$, therefore, neither of which can be divisible by p, there are $p-1$ terms, one of which must be divisible by p. If we knew nothing about $P+2$, we should say that the chance of its being $P+8$ was $1/(p-2)$. But, if we know that $P+2$ is prime, this number must be ruled out as a possible multiple of p, and the number divisible by p

must be one of the $p-3$ others. In other words, the chance of $P+8$ not being divisible by p will be $(p-3)/(p-1)$ as compared to 1 in the case of $P+3p$, which is certainly prime relative to p. Generalizing we may therefore say that the chance of $P+n$ being prime if we know that P and $P+2$ are primes is

$$\prod_{p|n,\, p>3} \frac{p-1}{p-3} = \prod_{p|n,\, p\geqslant 3} \left(1+\frac{2}{p-3}\right)$$

times as great as the chance of $P+n$ being prime when n has no prime divisors greater than 3.

The appearance of $p-3$ in the denominator in this formula of course indicates that, if P and $P+2$ are primes, the question whether a number is divisible by 3 is *ipso facto* decided.

Let us count the number of prime triplets in the neighbourhood of N of the form $P, P+2$, and $P+n$, where n is any number smaller than n_0 and n_0 is large compared with $\log N$. According to the prime number theorem there will be about $n_0/\log N$ primes between 0 and n_0 and therefore $n_0/\log N$ such triplets for each pair, P and $P+2$. But, as we have seen, the probability varies according to the number of different prime divisors of n. The fraction having any given value n will therefore be

$$\frac{\prod\limits_{p|n,\, p>5} \left(1+\frac{2}{p-3}\right)}{\sum\limits_{n=6}^{n=n_0} \prod\limits_{p|n,\, p>5} \left(1+\frac{2}{p-3}\right)}.$$

The denominator of course must be equal to the mean value of

$$\prod \left(1+\frac{2}{p-3}\right)$$

for all values of n between 4 and n_0 multiplied by the number of terms which is $\frac{1}{3}n_0$.

On the same lines as the above it is easy to show that S_3, the mean value of $\prod \left(1+\frac{2}{p-3}\right)$, converges to

$$S_3 = 1+2\sum_{p=5}^{\infty} \frac{1}{p(p-3)} + 4\sum_{\substack{p=5 \\ p_2>p_1}}^{\infty} \frac{1}{p_1(p_1-3)p_2(p_2-3)} + \ldots$$

$$= \prod_{p=5}^{\infty} \left(1+\frac{2}{p(p-3)}\right),$$

which is about $1{\cdot}386 \ldots$ for large values of n_0.

As has been shown, the number of pairs in an interval ΔN is

$$\frac{2}{S_2 \log N} \times \frac{\Delta N}{\log N}.$$

Each forms about $n_0/\log N$ prime triplets in which n in the term $P+n$ is less than n_0. The probability of finding one of them having the form $P, P+2, P+n$ is, as we have seen,

$$\frac{3}{S_3 n_0} \prod_{p|n,\, p>5} \left(1+\frac{2}{p-3}\right).$$

Hence, for large values of n_0, the number of triplets of the form $P, P+2, P+n$ in the interval ΔN should be

$$\Delta N \frac{6}{S_2 S_3 (\log N)^3} \prod_{p|n,\, p\geqslant 5} \left(1+\frac{2}{p-3}\right).$$

For $n = 6$ we find simply

$$F_3 = \frac{6\Delta N}{S_2 S_3 (\log N)^3} = \frac{2{\cdot}86...\Delta N}{(\log N)^3}.$$

Since all our arguments apply equally to positive and negative values of n, it is clear that the frequency of the triplets P, $P+2$, $P+6$ must be the same as that of the triplets P, $P+4$, $P+6$.

In Tables 3 and 3 (*a*), in which the sum of the frequencies of the

TABLE 3

$\Delta N.10^{-3}$	F_3 (calc.)	F_3 (obs.)
0–0·5	22·9	20
0·5–1	10·0	7
1–2	14·7	19
2–5	31·7	31
5–10	40·6	29
10–30	119·3	113
90–110	75·0	83
220–255	105·5	109
	419·7	411

TABLE 3 (*a*)

$\Delta N.10^{-5}$	F_3 (calc.)	F_3 (obs.)
1–2	339·8	333
2–3	298·4	312
3–4	275·1	312
4–5	259·3	248
5–6	247·6	244
6–7	238·4	248
7–8	230·9	220
8–9	224·6	200
9–10	219·2	215
	2333·3	2332

two types of triplets is used, observed and calculated values are compared. Our constant

$$\frac{1}{S_2 S_3} = \prod \frac{p^2(p-2)(p-3)}{(p-1)^3(p-2)} = \prod \left(1 - \frac{3p^2-1}{(p-1)^3}\right)$$

is the same as that found by Hardy and Littlewood from whose paper (loc. cit.) the number of triplets shown in Table 3 (*a*) is taken.

The numbers in the two groups are reasonably nearly equal. In Table 3 there are 202 of the type P, $P+2$, $P+6$ as against 206 of the type P, $P+4$, $P+6$. In Table 3 (*a*) the corresponding numbers are 1,138 and 1,194.

A similar though more complicated argument would give the formula for the frequency of quadruplets.*

* Though not strictly relevant it is perhaps interesting to point out that the number of primes of the form $4m^2+1$ can be derived by similar methods. The number of integers of this form in an interval ΔN is $\frac{1}{4}\Delta N/\sqrt{N}$; the probability of an odd integer being prime is $2/\log N$. Hence the number of primes of the form $4m^2+1$ less than n should be proportional to $\frac{1}{2}\mathrm{Li}\sqrt{n}$. To take account of the fact that 2 out of every p terms in the series $4m^2+1$ are divisible by p we must multiply this expression by

$$\prod_{p_1 > 5} \frac{p_1-2}{p_1-1},$$

p_1 being primes of the form $4\nu+1$. To allow for the fact that primes of the form $p_3 = 4\nu+3$ cannot divide $4m^2+1$ we must multiply by

$$\prod_{p_3 \geqslant 3} \frac{p_3}{p_3-1}.$$

If we do this, we arrive at the same formula as Hardy and Littlewood, namely

$$\frac{\mathrm{Li}\sqrt{n}}{2} \prod \left(1 - \frac{(-1)^{\frac{1}{2}(p-1)}}{p-1}\right).$$

Their formula for the frequency of prime pairs of the form $4m^2+1$ and $4m^2+3$ may be found on similar lines.

The number of representations of any even number as the sum of two primes (Goldbach's Theorem) may also be derived by probability arguments. If the number of different prime divisors of n is small compared with the number of primes less than n, the probability that $n-p_1$ will be a prime, n being a number between N and $N+\Delta N$, will tend as above to

$$\frac{2}{S_2 \log N} \prod_{p|n} \frac{p-1}{p-2} = \frac{2\,\mathrm{Li}\,n}{S_2 n} \prod_{p|n} \frac{p-1}{p-2}$$

if we average for all values of N less than n. The number of primes P_2 less than n is $\mathrm{Li}\,n$. Hence the probable number of occasions on which $P_1+P_2 = n$ tends to

$$\frac{2}{S_2}\frac{(\mathrm{Li}\,n)^2}{n} \prod_{p|n} \frac{p-1}{p-2} \to \frac{2}{S_2}\frac{n}{(\log n)^2} \prod_{p|n} \frac{p-1}{p-2},$$

as found by Hardy and Littlewood and by Staeckel.

The results derived from the assumption that the distribution of the primes may be treated as random are so well borne out that it seems worth while to carry our considerations a stage further and to see whether it is possible, albeit on very rough and ready lines, to calculate the frequency with which intervals between *neighbouring* primes occur.

Between P and $P+n$ there are $\frac{1}{2}(n-2)$ odd numbers. Normally, therefore, if we call the chance of $P+2\nu$ *not* being a prime $\alpha_{2\nu}$, ν being any whole number, the chance of no prime intervening between P and $P+n$, should be

$$\prod_{\nu=1}^{\nu=\frac{1}{2}(n-2)} \alpha_{2\nu}.$$

The probability w_n, therefore, that we shall find two neighbouring primes separated by an interval n should be equal to W_n, the inherent probability of this interval being found between two primes at all, multiplied by

$$\prod_{\nu=1}^{\nu=(\frac{1}{2}n-2)} \alpha_{2\nu},$$

which is the probability that all of the intervening numbers are composite.

It remains to find an expression for $\alpha_{2\nu}$. The first approximation is clearly given us by the prime-number theorem. If the chance of any number being a prime is $1/\log N$, the chance of an odd number being a prime is $2/\log N$, and the chance of an odd number *not* being a prime consequently $1-2/\log N$. Therefore we may expect to get something like the right result by putting

$$\alpha_{2\nu} = 1-\frac{2}{\log N}.*$$

* It should be pointed out that this rough and ready approach is not strictly consistent with the formula derived above for the frequency of triplets of primes of the form P, $P+2$, $P+6$ or P, $P+4$, $P+6$. If we take no account of intruding primes, there should be twice as many pairs of the form $P+6$ as of the form $P+2$ since $W_6 = 2W_2$. Hence, if the probability of $P+6$ being a neighbour of P is only $2W_2\left(1-\frac{2}{\log N}\right)$, it would follow that the frequency of triplets should be

$$\frac{W_2}{\log N}\times\frac{\Delta N}{\log N} = \frac{8}{S_2}\times\frac{\Delta N}{(\log N)^3}$$

instead of

$$\frac{12}{S_2 S_3}\times\frac{\Delta N}{(\log N)^3}$$

as derived more rigorously above,—a difference of about 8 per cent.

This assumption gives us the same value for the frequency of prime pairs as we derived above, namely $\frac{2\Delta N}{S_2(\log N)^2}$. If we designate the probability of intervals between neighbouring primes by small letters, clearly $W_2 = w_2$. Further, if $\alpha_{2\nu} = 1-2/\log N$,

$$w_n = w_2\left(1-\frac{2}{\log N}\right)^{\frac{1}{2}(n-2)} \prod_{p|n,p>3} \left(1+\frac{1}{p-2}\right).$$

Since P must be followed eventually by some prime, $\sum_2^\infty w_n = 1$. Substituting for the fluctuating expression $\prod \left(1+\frac{1}{p-2}\right)$ its mean value S_2, as we may by a well-known theorem,* we find

$$S_2 w_2 \sum \left\{1+\left(1-\frac{2}{\log N}\right)+\left(1-\frac{2}{\log N}\right)^2+\ldots\right\} = \frac{S_2 w_2}{2}\log N = 1,$$

which gives us for the probability of a prime P being followed by another $P+2$,

$$w_2 = \frac{2}{S_2 \log N},$$

as above. Obviously the crude expression

$$w_n = w_2\left(1-\frac{2}{\log N}\right)^{\frac{1}{2}(n-2)} \prod_{p|n,p>3} \left(1+\frac{1}{p-2}\right)$$

gives us also the right value for the mean value $\bar{n}$ of n. For, if we again take the average value of

$$\prod \left(1+\frac{1}{p-2}\right) = S_2,$$

it is clear that
$$\frac{\sum nw_n}{\sum w_n} = \log N.$$

In Table 1, F_4 the frequency of the interval 4 was given as well as F_2 and indeed was averaged with it in the last column. The reason is, of course, as we saw when considering prime triplets, that w_4 must be equal to w_2. For it is just as impossible that a prime should intervene between P and $P+4$ as that a prime should intervene between P and $P+2$. No prime can intervene between the two odd primes P and $P+2$ because the only intervening number is even—i.e.

* Hardy and Littlewood, *Proc. London Math. Soc.* 13 (1913), 174.

divisible by 2. No prime can intervene between P and $P+4$ because the only odd number between P and $P+4$ is $P+2$, and $P+2$ must

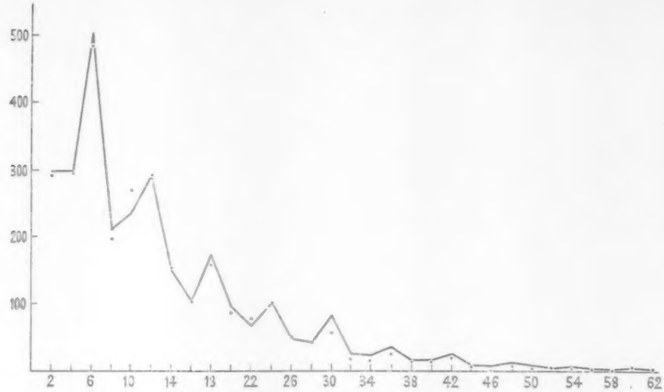

FIG. 2 (220,000 to 255,000)

In these diagrams (Figs. 2–4) the ordinates represent the frequencies of the various intervals which are plotted along the abscissae. The calculated frequencies are joined by straight lines; the observed frequencies are shown as separate dots.

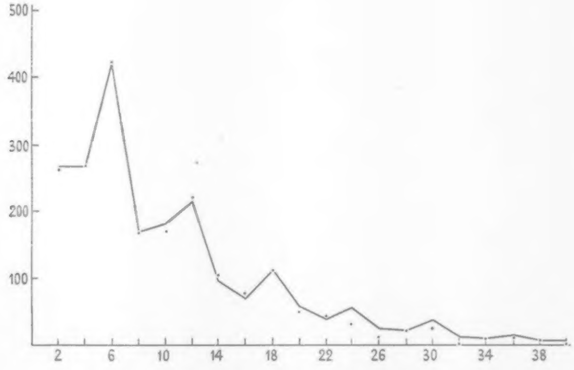

FIG. 3 (10,000 to 30,000)

be divisible by 3 if P and $P+4$ are primes. In Table 4 and Figs. 2, 3, and 4 observed and calculated frequencies of the intervals between neighbouring primes are compared. f_2 (calc.) and f_4 (calc.) are put

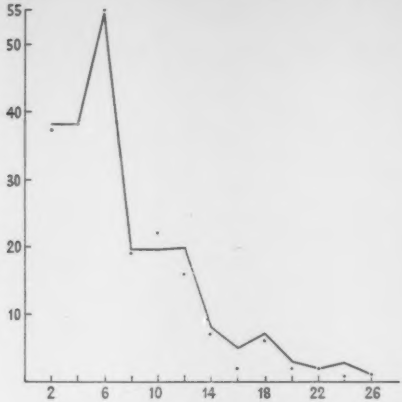

FIG. 4 (500 to 2,000)

equal to $\dfrac{2\Delta\pi}{S_2 \log N}$ as above; all the other terms are calculated according to the formula

$$f_n = f_2\left(1 - \frac{2}{\log N}\right)^{\frac{1}{2}(n-4)} \prod_{p|n,\, p\geq 3} \left(1 + \frac{1}{p-2}\right)$$

without any arbitrary constant. They show surprisingly good agreement between counts and calculation in the intervals 220,000 to 255,000, 90,000 to 110,000, 10,000 to 30,000, 2,000 to 10,000, and 500 to 2,000. This is the more remarkable since our formula for W_n ignored all second-order terms. In w_n we have introduced a term in $1/(\log N)^2$ although we had neglected the possibility of such a term in W_n which would of course appear in w_n. That the discrepancies are not larger seems to indicate that the second-order terms in W_n are relatively unimportant.

Although the observed and calculated values in these tables seldom differ by more than the expected statistical deviation, careful scrutiny does reveal a tendency to certain systematic discrepancies. In general the observed values seem to fall off rather more rapidly towards the end than the calculated. But there also seems to be a more specialized trend. Had we not realized that w_4 must be equal to w_2 and left the crude value

$$f_4 = f_2\left(1 - \frac{2}{\log N}\right),$$

we should have found observation systematically exceeding prediction

TABLE 4

n	220,000 to 255,000		90,000 to 110,000		10,000 to 30,000		2,000 to 10,000		500 to 2,000	
	f_n (calc.)	f_n (obs.)	f_n (calc.)	f_n (obs.)	f_n (calc.)	f_n (obs.)	f_n (calc.)	f_n (obs.)	f_n (calc.)	f_n (obs.)
2	299·5	295	199	198	267	262	140·5	144	38·3	37
4	299·5	299	199	206	267	267	140·5	141	38·3	38
6	502	485	329	312	426	426	218	219	55·2	55
8	210	197	135·8	135	170	168	84·5	74	19·8	19
10	235	271	149·6	174	181·3	167	76·2	91	19·2	22
12	296	292	185·5	157	216·8	220	101·3	84	20·7	16
14	148·5	153	91·8	97	94·1	103	47·2	43	8·9	7
16	103·8	103	63·4	64	69	75	30·5	32	5·3	2
18	174·2	159	104·7	103	110	110	47·3	34	7·7	6
20	97·2	87	57·7	48	58·8	48	24·5	13	3·7	2
22	68	79	39·7	43	39	42	15·8	14	2·2	2
24	102·5	101	59·1	57	56·1	30	22	14	3	1
26	47	50	26·6	24	24·5	12	9·3	3	1	
28	43·3	43	24·2	20	21·5	22	8·0	5		
30	80·7	57	44·3	32	38	27	13·6	11		
32	25·4	17	13·8	7	11·4	3	4·0	1		
34	22·7	14	12·1	13	9·7	9	3·3	2		
36	35·7	25	18·8	15	14·5	11	5·2	1		
38	15·8	14	8·2	5	6·1	0	2·0			
40	16·7	14	8·6	3	6·6	2				
42	25·1	19	12·7	5						
44	9·8	7	4·9	3						
46	7·7	2	4·0	5						
48	12·3	9	6·8	2						
50	9·3	3	3·2	1						
52	4·7	0	2·2	1						
54	7·3	3	3·4	3						
56	3·5	1	1·7							
58	2·6	2								
60	5·7	5		1						
62	1·9	0								

at $n = 4$. The other main systematic deviations, namely a tendency to an excessive number of intervals at $n = 10$, $n = 16$, and $n = 22$ are due to the same cause operating less obviously.

If we take divisibility by 3 into account, the number of places where primes can intervene is clearly substantially reduced. In the following table the second line shows l_2, the number of places in which primes might intrude having regard only to divisibility by 2, the third line the number if both 2 and 3 are considered.

$n =$	2	4	6	8	10	12	14	16	18	20	22	24
$l_2 =$	0	1	2	3	4	5	6	7	8	9	10	11
$l_3 =$	0	0	1	2	2	3	4	4	5	6	6	7

Clearly the number of possible intruders instead of rising steadily as in the second line halts always when $n = 6\nu+4$, ν being any whole number. It is easy to take this into account when $l_3 = 0$. The probability of $P+2$ being composite is 1 if P and $P+4$ are primes. But the answer is not so simple when l_3 halts at 2 instead of proceeding to 3. For the term $1-2/\log N$, which gives the probability that an odd number will be composite, must be modified. We cannot just substitute $3/\log N$ for $2/\log N$. True, this represents the probability of an odd number not divisible by 3 being prime. But this is not what we have got. We are dealing with the probability of a number between P and $P+n$ being prime, knowing P and $P+n$ to be prime. And to try to take this into account quantitatively is very complicated, as we have seen even in the comparatively straightforward case of prime triplets of the form P, $P+2$, $P+6$. Qualitatively, of course, it is easy to see what the result will be. Instead of the uniform exponential decrease there will be at these values of n a slowing up, leading to a slight excess of intervals between primes when $n = 6\nu+4$ over the number calculated by the simple expression. And this is what we observe.

This note does not claim to be more than a first step towards calculating the distribution of the intervals between primes by statistical methods. Clearly various other more intricate formulae could be derived by somewhat more elaborate processes and very likely better agreement could be achieved between calculation and observation. But even this crude approach gives results so closely in accord with observation that it would be difficult without an immense amount of statistical material to establish whether a genuine improvement had been made. As always in considerations of this kind it is questionable where the line should be drawn, i.e. where numbers are large enough to justify simple statistics.

It is, of course, possible that we have been fortunate in the selection of the relatively small sample it has been feasible to examine and that closer and more detailed study will reveal discrepancies not obvious in this preliminary investigation. But that these methods are capable of yielding interesting results seems to be proved.

Above all it seems remarkable that the formulae such as those for the frequency of pairs and triplets derived by much 'deeper' methods by Hardy and Littlewood based on an extension of Riemann's

hypothesis, should arise quite naturally in such a very simple way. It would appear interesting to examine whether and in what manner the Riemann hypothesis and Hardy and Littlewood's extension thereof are linked with the hypothesis that the distribution of the primes may be treated as random.

In conclusion I must express my thanks to Mr. J. Harvey for the care and skill with which he has carried out the counts which have enabled the comparisons to be made.

Note on the Previous Work

Hardy and Littlewood use their celebrated 'Partitio Numerorum' method to find likely asymptotic formulae for the number of pairs of primes $(P, P+2n)$ less than a large n and similar formulae for triplets, quadruplets, and so on. Their method is entirely distinct from that used in this paper and 'deeper'; the assumptions they make are very different in form, but fundamentally not dissimilar. Staeckel, on the other hand, uses probability arguments about as 'elementary' as in this note. His papers are long and, like Hardy and Littlewood's, deal also with Goldbach's Theorem; it is difficult to pick out exactly those parts which correspond to the work set out above, but the methods here used are much shorter than his.

Results equivalent to those found above for W_n were obtained by Hardy and Littlewood. They do not deal with the problem of w_n, though Staeckel gives certain preliminary considerations. In the triplets problem Hardy and Littlewood find a formula identical with that found above. Staeckel seems to have a result similar to theirs.

Hardy and Littlewood remark that a better comparison with numerical data is always given by using $\int\limits_2^n \frac{dx}{(\log x)^2}$ in place of $\frac{n}{(\log n)^2}$ in the formula for the number of prime-pairs less than n, and similarly in the other problems. The results derived by the methods used in this note appear naturally in terms of 'frequency', but they correspond, of course, to the integral form. It is presumably for this reason that the neglected second-order term in W_n turns out to be so small. A further advantage of the frequency form is that it enables numerical comparisons to be made over any interval; we do not have to start from 0 with our count.

So far no impression has been made on these problems by rigorous analysis. One of the best results so far *proved* in this direction appears to be that due to Buchstab, *Rec. Math.* (*Mat. Sbornik*), N.S. 10 (52), (1942), 87–91, who proves that, for any λ, there exist infinitely many primes p such that each prime factor of $P+2$ exceeds $(\log p)^\lambda$.

ELECTRICAL NOTES

By F. B. PIDDUCK (*Oxford*)

XII. ALTERNATING CURRENTS IN NETWORKS†

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THE mutual impedance of two open circuits at any frequency has been found by Murray.‡ Let P be a point on a circuit distinguished by the upper affix μ at a distance p from a fixed point on it, measured along the arc, and let the ends correspond to $p = p_1$ and $p = p_2$. Let Q similarly be a point on a circuit ν distant q from a fixed point on it, and let $PQ = r$. Let currents of frequency $\omega/2\pi$ flow in the circuits, and let the current at P be $I^\mu(p)\exp i\omega t$, where $I^\mu(p)$ is in general complex and the real part of the product is taken for a physical interpretation. Let $I^\mu(p)$ when split up into harmonics be of the form

$$\sum_{m=1}^{\infty} c_m^\mu I_m^\mu(p).$$

Murray found that the mth and nth harmonic currents in the circuits μ and ν have a mutual impedance given by

$$\frac{Z_{mn}^{\mu\nu}}{29\cdot 98} = \int\limits_{p_1}^{p_2}\int\limits_{q_1}^{q_2} \frac{\exp(-ikr)}{ikr}[I_m^{\mu *\prime}(p)I_n^{\nu\prime}(q) - k^2\cos\gamma I_m^{\mu *}(p)I_n^\nu(q)]\,dpdq,$$

where we have used rationalized metre-kilogram-second units and written k for $2\pi/\lambda$ and γ for the angle between the tangents at P and Q. To include both open and closed circuits, we have used the conjugate complex $I_m^{\mu *}(p)$ instead of $I_m^\mu(p)$ itself, and in this notation the tension applied to the circuit μ is

$$U_m^\mu = \int\limits_{p_1}^{p_2} E^\mu(p)I_m^{\mu *}(p)\,dp,$$

where $E^\mu(p)\exp i\omega t$ is the component of the electric intensity at P parallel to the tangent, in so far as it is applied from outside.

By considering $Z_{mn}^{\mu\mu}$ we find the self-impedance of a single circuit of thin wire, considered by Murray, and also extend the formula to networks. The double integral is not strictly a double line integral,

† This note is a continuation of Note IV, *Quart. J. of Math.* (Oxford), 2 (1931), 174.

‡ F. H. Murray, *American J. of Math.* 53 (1931), 873.

but a double surface integral, since the currents $I_m^\mu(p)$, $I_n^\nu(q)$ are distributed over the surfaces of the circuits. In calculating $Z_{mn}^{\mu\mu}$ we have to take the surface integrals twice over the same surface. Consider the contribution of the parts of the integral where r is small. It will not affect the argument much to take the current to be distributed uniformly over the cross-section, supposed circular. When r is small, $\exp(-ikr)/ikr$ is nearly equal to $1/ikr$, and the contribution is $\iint dSdS'/ikr$, where a constant has been dropped and dS, dS' are elements of surface. This is proportional to the potential energy of a uniform charge on a cylinder, which is equal to the mutual potential energy of two line charges, one in the line of centres and the other parallel to it in the surface of the wire. Thus $Z_{mn}^{\mu\mu}$ for a single wire is $Z_{mn}^{\mu\nu}$ for two such parallel geometrical lines, or filaments. Similar considerations apply to a network. Let the wires μ and ν have in common a piece of wire AB. Then in calculating $Z_{mn}^{\mu\nu}$ we must take the arc of one circuit there to be the line of centres and the arc of the other a parallel filament in the surface of the wire.

This theorem is more general than my former theorem in that the current need not be the same at all points, but less general in being restricted to currents lying wholly in the surface. It applies equally to closed or open circuits, except that $I_m^\mu(p)$ for a closed circuit need not vanish at any particular point of the wire. It is common to excite an antenna by attaching a feeder to two points on either side of the centre, and I have shown by these formulae that the best position of the points can be found approximately by theory.